CURVATURE FLOW OF COMPLETE CONVEX HYPERSURFACES IN HYPERBOLIC SPACE

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Abstract. We investigate the existence, convergence and uniqueness of modified general curvature flow (**MGCF**) of convex hypersurfaces in hyperbolic space with a prescribed asymptotic boundary.

1. Introduction

In this paper, we continue our study of modified curvature flow problems in hyperbolic space. Consider a complete (locally strictly) convex hypersurface in $\mathbb{H}^{n+1}$ with a prescribed asymptotic boundary Γ at infinity, whose principal curvatures satisfy $f(\kappa) > \sigma$, (e.g in our earlier work [LX10] section 8 we gave an example of such "good" initial surfaces.) and is given by an embedding $\mathbf{X}(0) : \Omega \to \mathbb{H}^{n+1}$, where $\Omega \subset \partial_\infty \mathbb{H}^{n+1}$. We consider the evolution of such embedding to produce a family of embeddings $\mathbf{X} : \Omega \times [0, T) \to \mathbb{H}^{n+1}$ satisfying the following equations

$$\begin{cases} \dot{\mathbf{X}} = (f(\kappa[\Sigma(t)]) - \sigma)\nu_H & (x,t) \in \Omega \times [0,T), \\ \mathbf{X}(0) = \Sigma_0 & (x,t) \in \partial\Omega \times \{0\}, \\ \mathbf{X} = \Gamma & (x,t) \in \partial\Omega \times [0,T), \end{cases} \tag{1.1}$$

where $\kappa[\Sigma(t)] = (\kappa_1, \cdots, \kappa_n)$ denotes the hyperbolic principal curvatures of $\Sigma(t)$, $\sigma \in (0,1)$ is a constant and ν_H denotes the outward unit normal of $\Sigma(t)$ with respect to the hyperbolic metric.

In this paper, we shall use the half-space model,

$$\mathbb{H}^{n+1} = \{(x, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} > 0\}$$

equipped with the hyperbolic metric

$$ds^2 = \frac{\sum_{i=1}^{n+1} dx_i^2}{x_{n+1}^2}. \tag{1.2}$$

One identifies the hyperplane $\{x_{n+1} = 0\} = \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$ as infinity of $\mathbb{H}^{n+1}$, denoted by $\partial_\infty \mathbb{H}^{n+1}$. For convenience we say Σ has compact asymptotic boundary if $\partial\Sigma \subset \partial_\infty \mathbb{H}^{n+1}$ is compact with respect to the Euclidean metric in $\mathbb{R}^n$.

The function f is assumed to satisfy the following fundamental structure conditions:

$$f_i(\lambda) \equiv \frac{\partial f(\lambda)}{\partial \lambda_i} > 0 \ \text{ in } K, \ \ 1 \le i \le n, \tag{1.3}$$

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$$f \text{ is a concave function in } K, \tag{1.4}$$

and

$$f > 0 \text{ in } K, \quad f = 0 \text{ on } \partial K, \tag{1.5}$$

where $K \subset \mathbb{R}^n$ is an open symmetric convex cone defined as following

$$K := K_n^+ := \{\lambda \in \mathbb{R}^n : \text{each component } \lambda_i > 0\}. \tag{1.6}$$

In addition, we shall assume that f is normalized

$$f(1, \cdots, 1) = 1 \tag{1.7}$$

and satisfies more technical assumptions

$$f \text{ is homogeneous of degree one.} \tag{1.8}$$

Moreover,

$$\lim_{R \to +\infty} f(\lambda_1, \cdots, \lambda_{n-1}, \lambda_n + R) \geq 1 + \epsilon_0 \quad \text{uniformly in } B_{\delta_0}(\mathbf{1}) \tag{1.9}$$

for some fixed $\epsilon_0 > 0$ and $\delta_0 > 0$, where $B_{\delta_0}(\mathbf{1})$ is the ball of radius δ_0 centered at $\mathbf{1} = (1, \cdots, 1) \in \mathbb{R}^n$.

As shown in [GS10], an example of the function satisfies all assumptions above is given by $f = (H_n/H_l)^{\frac{1}{n-l}}$, $0 \leq l < n$, defined in K, where H_l is the normalized $l-th$ elementary symmetric polynomial. (e.g., $H_0 = 1$, $H_1 = H$, $H_n = K$ the extrinsic Gauss curvature.)

Since f is symmetric, by (1.4), (1.7) and (1.8) we have

$$f(\lambda) \leq f(\mathbf{1}) + \sum f_i(\mathbf{1})(\lambda_i - 1) = \sum f_i(\mathbf{1})\lambda_i = \frac{1}{n}\sum \lambda_i \text{ in } K \tag{1.10}$$

and

$$\sum f_i(\lambda) = f(\lambda) + \sum f_i(\lambda)(1 - \lambda_i) \geq f(\mathbf{1}) = 1 \text{ in } K. \tag{1.11}$$

In this paper, we always assume the initial surfaces Σ_0 to be connected and orientable, and $\Sigma(t) = \{\mathbf{X} := (x, u(x,t)) \mid (x,t) \in \Omega \times [0,T),\ x_{n+1} = u(x,t)\}$ to be the flow surfaces with $\mathbf{X} = (x, u(x,t))$ satisfying the flow equation (1.1). If Σ is a complete hypersurface in $\mathbb{H}^{n+1}$ with compact asymptotic boundary at infinity, then the normal vector field of Σ is always chosen to be the one pointing to the unique unbounded region in $\mathbb{R}^{n+1}_+/\Sigma$, and both Euclidean and hyperbolic principal curvature of Σ are calculated with respect to this normal vector field.

We shall take $\Gamma = \partial\Omega$, where $\Omega \subset \mathbb{R}^n$ is a smooth domain and seek a family of hypersurfaces $\Sigma(t)$ as a graph of function $u(x,t)$ with boundary Γ. Then the coordinate vector fields and upper unit normal are given by

$$\mathbf{X}_i = e_i + u_i e_{n+1}, \ \nu_H = u\nu = u\frac{(-u_i e_i + e_{n+1})}{w},$$

where through out this paper, $w = \sqrt{1 + |\nabla u|^2}$, e_{n+1} is the unit vector in the positive x_{n+1} direction in $\mathbb{R}^{n+1}$, ν_H denotes the hyperbolic unit normal, and ν denotes the Euclidean unit normal.

Note that by equation (1.1)

$$\left\langle \dot{\mathbf{X}}, \nu_H \right\rangle_H = f - \sigma,$$

which is equivalent to

$$\left\langle \frac{\partial}{\partial t}(x, u(x,t)), \nu_H \right\rangle_H = f - \sigma,$$

from here we can derive that the height function u satisfies equation

$$u_t = (f - \sigma)uw. \tag{1.12}$$

So problem (1.1) then reduces to the Dirichlet problem for a fully nonlinear second order parabolic equation

$$\begin{cases} u_t = uw(f - \sigma) & \text{on } \Omega \times [0, T)\,, \\ u(x, 0) = u_0 & \text{on } \Omega \times \{0\}\,, \\ u(x, t) = 0 & \text{on } \partial\Omega \times [0, T)\,. \end{cases} \tag{1.13}$$

In this paper, we shall focus on proving the long time existence of the modified general curvature flow (**MGCF**) of complete embedded hypersurfaces with initial surface whose principal curvatures satisfy $f(\kappa) > \sigma$ everywhere; furthermore, we shall also prove the uniqueness under additional assumptions.

To begin with, I'd like to state the following beautiful result of [GSZ09]

Theorem 1.1. *Let Σ be a complete locally strictly convex C^2 hypersurface in $\mathbb{H}^{n+1}$ with compact asymptotic boundary at infinity. Then Σ is the vertical graph of a function $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$, $u > 0$ in Ω and $u = 0$ on $\partial\Omega$, for some domain $\Omega \subset \mathbb{R}^n$:*

$$\Sigma = \{(x, u) \in \mathbb{R}^{n+1}_+ : x \in \Omega\}$$

such that

$$\{\delta_{ij} + u_i u_j + u u_{ij}\} > 0 \ \text{ in } \Omega. \tag{1.14}$$

That is, the function $u^2 + |x|^2$ is strictly convex.

According to Theorem 1.1, our assumption that $\Sigma(t)$ is a graph is completely general and the asymptotic boundary Γ must be the boundary of some bounded domain Ω in $\mathbb{R}^n$.

We seek solution of equation (1.13) satisfying (1.14) for all $t \in [0, T)$. (We will see in section 4 that when the initial surface of the MGCF under certain restriction then the solution of (1.13) must satisfy (1.14).) Following the literature we call such solutions *admissible*. By [CNS85] condition (1.3) implies that equation (1.13) is parabolic for admissible solutions.

The main result of this paper may be stated as follows.

Theorem 1.2. *Let $\Gamma = \partial\Omega \times \{0\} \subset \mathbb{R}^{n+1}$ where Ω is a bounded smooth domain in $\mathbb{R}^n$. Suppose that $\sigma \in (0, 1)$ and that f satisfies conditions (1.3)-(1.9) with $K = K_n^+$.*

Furthermore, let $\Sigma_0 = \{(x, u_0(x)) \mid u_0 \in C^\infty(\Omega) \cap C^{1+1}(\overline{\Omega})\}$ *be a complete locally strictly convex hypersurface with* $\partial\Sigma_0 = \Gamma$ *and* $f(\kappa[\Sigma_0])$ *greater than* σ*, then there exists a solution* $\Sigma(t),\ t \in [0,\infty)$*, to the MGCF (1.1) with uniformly bounded principal curvatures*

$$|\kappa[\Sigma(t)]| \leq C \quad \text{on } \Sigma(t), \text{ for all } t \in [0,\infty). \tag{1.15}$$

Moreover, $\Sigma(t) = \{(x, u(x,t)) \mid (x,t) \in \Omega \times [0,\infty)\}$ *is the flow surface of an admissible solution* $u(x,t) \in C^\infty(\Omega \times (0,\infty)) \cap W_p^{2,1}(\Omega \times [0,\infty))$ [1] *of the Dirichlet problem (1.13), where* $p > 4$*. Furthermore, for any fixed* $t > 0$*, we have* $u^2(x,t) \in C^\infty(\Omega) \cap C^{1+1}(\overline{\Omega})$ *and*

$$u|D^2u| \leq C \quad \text{in } \Omega, \tag{1.16}$$

$$\sqrt{1+|Du|^2} \leq C \quad \text{in } \Omega, \tag{1.17}$$

where C *is some constant independent of* t*. In addition, if*

$$\sum f_i > \sum \lambda_i^2 f_i \quad \text{in } K \cap \{0 < f < 1\}, \tag{1.18}$$

then as $t \to \infty$*,* $u(t)$ *converges uniformly to a function* $\tilde{u} \in C^\infty(\Omega) \cap C^1(\overline{\Omega})$*, such that* $\Sigma_\infty = \{(x,\tilde{u}) \in \mathbb{R}^{n+1}, x \in \Omega\}$ *is a unique complete locally strictly convex surface satisfies* $f(\kappa[\Sigma_\infty]) = \sigma$ *in* $\mathbb{H}^{n+1}$*.*

Due to the degeneracy of equation (1.13) when $u = 0$, it is very natural to consider the approximate modified general curvature flow (**AMGCF**) problem. Instead of $u = 0$ on $\partial\Omega$ one assumes $u = \epsilon$ on $\partial\Omega$, ϵ is small enough. So the equations become,

$$\begin{cases} u_t = uw(f-\sigma) & \text{on } \Omega \times [0,T), \\ u(x,0) = u_0^\epsilon & \text{on } \Omega \times \{0\}, \\ u(x,t) = \epsilon & \text{on } \partial\Omega \times [0,T). \end{cases} \tag{1.19}$$

where $u_0^\epsilon = u_0 + \epsilon$ and $\Sigma_0^\epsilon = \{(x, u_0^\epsilon) | x \in \Omega\}$ satisfies $f(\kappa[\Sigma_0^\epsilon]) > \sigma,\ \forall x \in \Omega$.

Theorem 1.3. *Let* Ω *be a bounded smooth domain in* $\mathbb{R}^n$ *and* $\sigma \in (0,1)$*. Suppose* f *satisfies (1.3)-(1.9) with* $K = K_n^+$*. Then for any* $\epsilon > 0$ *sufficiently small, there exists an admissible solution* $u^\epsilon \in C^\infty(\overline{\Omega} \times (0,\infty))$ *of the Dirichlet Problem (1.19). Moreover,* u^ϵ *satisfies the a priori estimates*

$$\sqrt{1+|Du^\epsilon|^2} \leq \frac{1}{\sigma} + C\epsilon, \quad u^\epsilon|D^2u^\epsilon| \leq C \quad \text{on } \partial\Omega \times [0,\infty), \tag{1.20}$$

and

$$u^\epsilon|D^2u^\epsilon| \leq C(t,\epsilon) \quad \text{in } \Omega \times [0,\infty). \tag{1.21}$$

[1] $W_p^{2,1}$ is the space of function f such that the norms

$$\|f\|_{w_p^{2,1}} = \|f\|_{Lp} + \left\|\frac{\partial f}{\partial t}\right\|_{Lp} + \sum_{i=1}^n \left\|\frac{\partial f}{\partial x_i}\right\|_{Lp} + \sum_{i,j=1}^n \left\|\frac{\partial^2 f}{\partial x_i \partial x_j}\right\|_{Lp}$$

are finite.

In particular, $C(t, \epsilon)$ depends exponentially on time t.

Remark 1.4. The a priori estimates (1.20) will be proved in section 5 and 6, while estimate (1.21) can be proved by combining Lemma 7.2 and equation (7.12) then use standard maximum principle for parabolic equation.

The main technical difficulty in proving Theorem 1.2 is that we can not use the estimates (1.21) to pass to the limit. We overcome this difficulty by proving a maximum principle for the largest hyperbolic principal curvature.

Theorem 1.5. *Let Ω be a bounded smooth domain in $\mathbb{R}^n$ and $\sigma \in (0,1)$. Suppose f satisfies (1.3)-(1.9) with $K = K_n^+$. Then for any admissible solution u^ϵ of the Dirichlet problem (1.19),*

$$u^\epsilon |D^2 u^\epsilon| \leq C(1 + \max_{\partial\Omega\times[0,\infty)} u^\epsilon |D^2 u^\epsilon|) \;\; \text{in } \Omega \times [0,\infty), \tag{1.22}$$

where C is independent of ϵ and t.

By applying Theorem 1.5 to Theorem 1.3, one can see that the hyperbolic curvatures of the admissible solution u^ϵ are uniformly bounded from above. Later we will also show that, if our initial surface satisfies $f(\kappa[\Sigma_0]) > \sigma$ then $f > \sigma$ during the flow process. In particular,

Theorem 1.6. *Suppose f satisfies (1.3)-(1.9) with $K = K_n^+$, and $u^\epsilon(x,t)$ is an admissible solution of the Dirichlet problem (1.19), and in addition*

$$f(\kappa[\Sigma_0^\epsilon]) > \sigma. \tag{1.23}$$

Then we have

$$f(\kappa[\Sigma^\epsilon(t)]) > \sigma \;\; \forall t \in [0,T). \tag{1.24}$$

Thus one can conclude that the hyperbolic curvatures admit a uniform positive lower bound, so by the interior estimates of Evans and Krylov, we obtain a uniform $C^{2,\alpha}$ estimates for any compact subdomain of Ω. Then the proof of Theorem 1.2 becomes routine.

The paper is organized as follows. In section 2 we establish some basic identities for hypersurfaces in $\mathbb{H}^{n+1}$. Section 3 contains some essential identities and evolution equations which will be used later. The preserving of convexity will be proved in section 4. Section 5 contains a global gradient estimate, while in sections 6 and 7 we prove the boundary and global estimates for the second derivative of u respectively. Finally in sections 8 and 9, we discuss the convergence and uniqueness of the MGCF.

2. Formulas for hyperbolic principal curvatures

2.1. **Formulas on hypersurfaces.** We will compare the induced hyperbolic and Euclidean metrics and derive some basic identities on a hypersurface.

Let Σ be a hypersurface in $\mathbb{H}^{n+1}$. We shall use g, and ∇ to denote the induced hyperbolic metric and Levi-Civita connections on Σ, respectively. Since Σ also can be viewed as a submanifold of $\mathbb{R}^{n+1}$, we shall usually distinguish a geodesic quantity with respect to Euclidean metric by adding a 'tilde' over the corresponding hyperbolic quantity. For instance, $\tilde{g}$ denotes the induced metric on Σ from $\mathbb{R}^{n+1}$, and $\tilde{\nabla}$ is its Levi-Civita connection.

Let $(z_1, \cdots, z_n)$ be local coordinates and

$$\tau_i = \frac{\partial}{\partial z_i}, \ \ i = 1, \cdots, n.$$

The hyperbolic and Euclidean metrics of Σ are given by

$$g_{ij} = \langle \tau_i, \tau_j \rangle_H, \ \ \tilde{g}_{ij} = \tau_i \cdot \tau_j = u^2 g_{ij}, \tag{2.1}$$

while the second fundamental forms are

$$\begin{aligned} h_{ij} &= \langle D_{\tau_i}\tau_j, \nu_H \rangle_H = - \langle D_{\tau_i}\nu_H, \tau_j \rangle_H, \\ \tilde{h}_{ij} &= \nu \cdot \tilde{D}_{\tau_i}\tau_j = -\tau_j \cdot \tilde{D}_{\tau_i}\nu, \end{aligned} \tag{2.2}$$

where D and $\tilde{D}$ denote the Levi-Civita connection of $\mathbb{H}^{n+1}$ and $\mathbb{R}^{n+1}$, respectively. The following relations are well known (see equation(1.5),(1.6) of [GS08]):

$$h_{ij} = \frac{1}{u}\tilde{h}_{ij} + \frac{\nu^{n+1}}{u^2}\tilde{g}_{ij}. \tag{2.3}$$

$$\kappa_i = u\tilde{\kappa}_i + \nu^{n+1}, \ \ i = 1, \cdots, n, \tag{2.4}$$

where $\nu^{n+1} = \nu \cdot e_{n+1} = \frac{1}{w}$.

The Christoffel symbols are related by formula

$$\Gamma_{ij}^k = \tilde{\Gamma}_{ij}^k - \frac{1}{u}(u_i\delta_{kj} + u_j\delta_{ik} - \tilde{g}^{kl}u_l\tilde{g}_{ij}). \tag{2.5}$$

It follows that for $v \in C^2(\Sigma)$

$$\nabla_{ij}v = v_{ij} - \Gamma_{ij}^k v_k = \tilde{\nabla}_{ij}v + \frac{1}{u}(u_iv_j + u_jv_i - \tilde{g}^{kl}u_kv_l\tilde{g}_{ij}) \tag{2.6}$$

where and in the sequel (if no additional explanation)

$$v_i = \frac{\partial v}{\partial x_i}, \ v_{ij} = \frac{\partial^2 v}{\partial x_i \partial x_j}, \ etc.$$

In particular,

$$\nabla_{ij}u = \tilde{\nabla}_{ij}u + \frac{2u_iu_j}{u} - \frac{1}{u}\tilde{g}^{kl}u_ku_l\tilde{g}_{ij}. \tag{2.7}$$

Moreover in $\mathbb{R}^{n+1}$,

$$\tilde{g}^{kl}u_ku_l = |\tilde{\nabla}u|^2 = 1 - (\nu^{n+1})^2 \tag{2.8}$$

$$\tilde{\nabla}_{ij}u = \tilde{h}_{ij}\nu^{n+1}. \tag{2.9}$$

We note that all formulas above still hold for general local frame $\tau_1, \cdots, \tau_n$. In particular, if $\tau_1, \cdots, \tau_n$ are orthonormal in the hyperbolic metric, then $g_{ij} = \delta_{ij}$ and $\tilde{g}_{ij} = u^2\delta_{ij}$.

We now consider equation (1.1) on Σ. For K as in section 1, let $\mathcal{A}$ be the vector space of $n \times n$ matrices and

$$\mathcal{A}_K = \{A = \{a_{ij}\} \in \mathcal{A} : \lambda(A) \in K\},$$

where $\lambda(A) = (\lambda_1, \cdots, \lambda_n)$ denotes the eigenvalues of A. Let F be the function defined by

$$F(A) = f(\lambda(A)), \quad A \in \mathcal{A}_K \tag{2.10}$$

and denote

$$F^{ij}(A) = \frac{\partial F}{\partial a_{ij}}(A), \quad F^{ij,kl}(A) = \frac{\partial^2 F}{\partial a_{ij} \partial a_{kl}}(A). \tag{2.11}$$

Since $F(A)$ depends only on the eigenvalues of A, if A is symmetric then so is the matrix $\left\{F^{ij}(A)\right\}$. Moreover,

$$F^{ij}(A) = f_i \delta_{ij}$$

when A is diagonal, and

$$F^{ij}(A) a_{ij} = \sum f_i(\lambda(A)) \lambda_i = F(A), \tag{2.12}$$

$$F^{ij}(A) a_{ik} a_{jk} = \sum f_i(\lambda(A)) \lambda_i^2. \tag{2.13}$$

Equation (1.13) can therefore be rewritten in a local frame $\tau_1, \cdots, \tau_n$ in the form

$$\begin{cases} u_t = uw(F(A[\Sigma]) - \sigma) & (x,t) \in \Omega \times [0,T), \\ u(x,0) = u_0 & (x,t) \in \Omega \times \{0\}, \\ u(x,t) = 0 & (x,t) \in \partial\Omega \times [0,T), \end{cases} \tag{2.14}$$

where $A[\Sigma] = \left\{g^{ik} h_{kj}\right\}$.

2.2. **Vertical graphs.** Suppose Σ is locally represented as the graph of a function $u \in C^2(\Omega)$, $u > 0$, in a domain $\Omega \subset \mathbb{R}^n$:

$$\Sigma = \{(x, u(x)) \in \mathbb{R}^{n+1} : x \in \Omega\}.$$

In this case we take ν to be the upward (Euclidean) unit normal vector field to Σ :

$$\nu = \left(-\frac{Du}{w}, \frac{1}{w}\right), \; w = \sqrt{1 + |Du|^2}.$$

The Euclidean metric and second fundamental form of Σ are given respectively by

$$\tilde{g}_{ij} = \delta_{ij} + u_i u_j,$$

and

$$\tilde{h}_{ij} = \frac{u_{ij}}{w}.$$

According to [CNS86], the Euclidean principal curvature $\tilde{\kappa}[\Sigma]$ are the eigenvalues of symmetric matrix $\tilde{A}[u] = [\tilde{a}_{ij}]$:

$$\tilde{a}_{ij} := \frac{1}{w} \gamma^{ik} u_{kl} \gamma^{lj}, \tag{2.15}$$

where

$$\gamma^{ij} = \delta_{ij} - \frac{u_i u_j}{w(1+w)}.$$

Note that the matrix $\{\gamma^{ij}\}$ is invertible with the inverse

$$\gamma_{ij} = \delta_{ij} + \frac{u_i u_j}{1+w}$$

which is the square root of $\{\tilde{g}_{ij}\}$, i.e., $\gamma_{ik}\gamma_{kj} = \tilde{g}_{ij}$. From (2.4) we see that the hyperbolic principal curvatures $\kappa[u]$ of Σ are eigenvalues of the matrix $A[u] = \{a_{ij}[u]\}$:

$$a_{ij} := \frac{1}{w}\left(\delta_{ij} + u\gamma^{ik}u_{kl}\gamma^{lj}\right). \tag{2.16}$$

When Σ is a vertical graph we can also define $F(A[\Sigma]) = F(A[u])$.

3. Short time existence and Evolution equations

3.1. **Short time existence.** In order to prove a global existence for the Dirichlet problem (1.19), we need to start with a short time existence theorem. Though this theorem is standard, for completeness we state it as follows:

Theorem 3.1. *Let $G(D^2u, Du, u)$ be a nonlinear operator, which is smooth with respect to u, Du and D^2u. Suppose that G is defined for function u belonging to an open set $\Lambda \subset C^2(\Omega)$ and G is elliptic for any $u \in \Lambda$, i.e., $G^{ij} > 0$. Then the initial value problem*

$$\begin{cases} u_t = G(D^2u, Du, u) & \text{on } \Omega \times [0, T), \\ u(x, 0) = u_0 & \text{on } \Omega \times \{0\}, \\ u(x, t) = u_0|_{\partial\Omega} & \text{on } \partial\Omega \times [0, T), \end{cases} \tag{3.1}$$

has a unique solution u for $T = \epsilon > 0$ small enough. Furthermore, u is smooth except for the corner, when $u_0 \in \Lambda$ is of class $C^\infty(\overline{\Omega})$.

3.2. **Evolution equations for some geometric quantities.** In this subsection, we will compute the evolution equations for some affine geometric quantities. Before we start, need to point out that in this section for $v \in C^2(\Sigma)$, we denote $v_i = \tilde{\nabla}_i v$, $v_{ij} = \tilde{\nabla}_{ij} v$, etc.

Lemma 3.2. *(Evolution of the metrics). The metric g_{ij} and $\tilde{g}_{ij}$ of $\Sigma(t)$ satisfies the evolution equations*

$$\dot{g}_{ij} = -2u^{-2}\tilde{g}_{ij}(F - \sigma)w - 2u^{-1}(F - \sigma)\tilde{h}_{ij}, \tag{3.2}$$

and

$$\dot{\tilde{g}}_{ij} = -2(F - \sigma)u\tilde{h}_{ij}. \tag{3.3}$$

Proof. Since $\tilde{g}_{ij} = \tau_i \cdot \tau_j$,

$$\begin{aligned} \frac{\partial}{\partial t}\tilde{g}_{ij} &= 2\left\langle \tilde{D}_{\tau_i}\dot{X}, \tilde{D}_{\tau_j}X \right\rangle \\ &= 2\left\langle \tilde{D}_{\tau_i}[(F-\sigma)u\nu], \tau_j \right\rangle \\ &= 2(F-\sigma)u\left\langle \tilde{D}_{\tau_i}\nu, \tau_j \right\rangle \\ &= -2(F-\sigma)u\tilde{h}_{ij}. \end{aligned}$$

Differentiating equation (2.1) with respect to t we get

$$\begin{aligned} \frac{\partial}{\partial t}g_{ij} &= -2u^{-3}\tilde{g}_{ij}u_t + u^{-2}\dot{\tilde{g}}_{ij} \\ &= -2u^{-3}\tilde{g}_{ij}(F-\sigma)uw - 2u^{-2}(F-\sigma)u\tilde{h}_{ij} \\ &= -2u^{-2}\tilde{g}_{ij}(F-\sigma)w - 2u^{-1}(F-\sigma)\tilde{h}_{ij}. \end{aligned}$$

□

Lemma 3.3. *(Evolution of the normal). The normal vector evolves according to*

$$\dot{\nu} = -\tilde{g}^{ij}[(F-\sigma)u]_i\tau_j, \tag{3.4}$$

moreover,

$$\dot{\nu}^{n+1} = -\tilde{g}^{ij}[(F-\sigma)u]_iu_j. \tag{3.5}$$

Proof. Since ν is the unit normal vector of Σ, we have $\dot{\nu} \in T(\Sigma)$. Furthermore, differentiating

$$\langle \nu, \tau_i \rangle = \left\langle \nu, \tilde{D}_{\tau_i}X \right\rangle = 0,$$

with respect to t we deduce

$$\begin{aligned} \langle \dot{\nu}, \tau_i \rangle &= -\left\langle \nu, \tilde{D}_{\tau_i}[(F-\sigma)u\nu] \right\rangle \\ &= -\langle \nu, [(F-\sigma)u]_i\nu \rangle \\ &= -[(F-\sigma)u]_i, \end{aligned}$$

so we have

$$\dot{\nu} = -\tilde{g}^{ij}[(F-\sigma)u]_i\tau_j.$$

Thus (3.5) follows directly from

$$\dot{\nu}^{n+1} = \langle \dot{\nu}, e_{n+1} \rangle \text{ and } u_j = \tau_j \cdot e_{n+1}.$$

□

Lemma 3.4. *(Evolution of the second fundamental form). The second fundamental form evolves according to*

$$\dot{\tilde{h}}_i^l = [(F-\sigma)u]_i^l + u(F-\sigma)\tilde{h}_i^k\tilde{h}_k^l, \tag{3.6}$$

$$\dot{\tilde{h}}_{ij} = [(F-\sigma)u]_{ij} - u(F-\sigma)\tilde{h}_i^k\tilde{h}_{kj}, \tag{3.7}$$

and

$$\begin{aligned}\dot{h}_{ij} &= \frac{1}{u}\{[(F-\sigma)u]_{ij} - u(F-\sigma)\tilde{h}_i^k\tilde{h}_{kj}\} - \frac{\tilde{h}_{ij}}{u}w(F-\sigma)\\ &- \{\tilde{g}^{kl}[u(F-\sigma)]_k u_l\}\frac{\tilde{g}_{ij}}{u^2} - 2\frac{(F-\sigma)\nu^{n+1}}{u}\tilde{h}_{ij} - 2\frac{\tilde{g}_{ij}}{u^2}(F-\sigma).\end{aligned} \tag{3.8}$$

Proof. Differentiating (3.4) with respect to τ_i we get

$$\frac{\partial}{\partial t}\nu_i = -\tilde{g}^{kl}[(F-\sigma)u]_{ki}\tau_l - \tilde{g}^{kl}[(F-\sigma)u]_k\tilde{D}_{\tau_i}\tau_l.$$

On the other hand, in view of the Weingarten Equation

$$\nu_i = -\tilde{g}^{kl}\tilde{h}_{li}\tau_k \Rightarrow \dot{\nu}_i = -\dot{\tilde{h}}_i^k\tau_k - \tilde{h}_i^k\tilde{D}_{\tau_k}\dot{X},$$

where $\tilde{h}_i^k = \tilde{g}^{kl}\tilde{h}_{li}$ is a mixed tensor, multiply by τ_j we get

$$-\dot{\tilde{h}}_i^k\tilde{g}_{kj} - \tilde{h}_i^k\left\langle \tilde{D}_{\tau_k}\dot{X}, \tau_j\right\rangle = -\tilde{g}^{kl}[(F-\sigma)u]_{ki}\tilde{g}_{lj}.$$

Therefore

$$\begin{aligned}\dot{\tilde{h}}_i^k\tilde{g}_{kj} &= \tilde{g}^{kl}[(F-\sigma)u]_{ki}\tilde{g}_{lj} - \tilde{h}_i^k u(F-\sigma)\left\langle \tilde{D}_{\tau_k}\nu, \tau_j\right\rangle\\ &= [(F-\sigma)u]_{ij} + u(F-\sigma)\tilde{h}_i^k\tilde{h}_{kj}.\end{aligned}$$

Multiplying the resulting equation with $\tilde{g}^{jl}$

$$\dot{\tilde{h}}_i^l = [(F-\sigma)u]_i^l + u(F-\sigma)\tilde{h}_i^k\tilde{h}_k^l. \tag{3.9}$$

Moreover, since $\tilde{h}_{ij} = \tilde{h}_i^l\tilde{g}_{lj}$, differentiating it with respect to t and use equation (3.3) get

$$\begin{aligned}\dot{\tilde{h}}_{ij} &= \dot{\tilde{h}}_i^l\tilde{g}_{lj} + \tilde{h}_i^l\dot{\tilde{g}}_{lj}\\ &= [(F-\sigma)u]_i^l\tilde{g}_{lj} + u(F-\sigma)\tilde{h}_i^k\tilde{h}_k^l\tilde{g}_{lj} + \tilde{h}_i^l[-2(F-\sigma)u\tilde{h}_{lj}]\\ &= [(F-\sigma)u]_{ij} - u(F-\sigma)\tilde{h}_i^k\tilde{h}_{kj}.\end{aligned}$$

Finally by differentiating equation (2.3) with respect to t, we have

$$\begin{aligned}\frac{\partial}{\partial t}h_{ij} &= \frac{1}{u}\dot{\tilde{h}}_{ij} - \frac{\tilde{h}_{ij}}{u^2}u_t + \frac{\tilde{g}_{ij}}{u^2}\dot{\nu}^{n+1} + \frac{\nu^{n+1}}{u^2}\dot{\tilde{g}}_{ij} - 2\frac{\nu^{n+1}\tilde{g}_{ij}}{u^3}u_t\\ &= \frac{1}{u}\{[(F-\sigma)u]_{ij} - u(F-\sigma)\tilde{h}_i^k\tilde{h}_{kj}\} - \frac{\tilde{h}_{ij}}{u}w(F-\sigma)\\ &+ \frac{\tilde{g}_{ij}}{u^2}\{-\tilde{g}^{kl}[u(F-\sigma)]_k u_l\} + \frac{\nu^{n+1}}{u^2}[-2(F-\sigma)u\tilde{h}_{ij}] - 2\frac{\nu^{n+1}\tilde{g}_{ij}}{u^3}uw(F-\sigma)\\ &= \frac{1}{u}\{[(F-\sigma)u]_{ij} - u(F-\sigma)\tilde{h}_i^k\tilde{h}_{kj}\} - \frac{\tilde{h}_{ij}}{u}w(F-\sigma)\\ &- \{\tilde{g}^{kl}[u(F-\sigma)]_k u_l\}\frac{\tilde{g}_{ij}}{u^2} - 2\frac{(F-\sigma)\nu^{n+1}}{u}\tilde{h}_{ij} - 2\frac{\tilde{g}_{ij}}{u^2}(F-\sigma).\end{aligned} \tag{3.10}$$

□

Lemma 3.5. *(Evolution of F) The term F evolves according to the equation*

$$\begin{aligned} F_t = {} & uF^{ij}[(F-\sigma)u]_i^j + (F-\sigma)\left[\sum f_s\kappa_s^2 - 2\nu^{n+1}F + (\nu^{n+1})^2\sum f_s\right] \\ & + w(F-\sigma)\left(F - \nu^{n+1}\sum f_s\right) - [(F-\sigma)u]_i u^i \sum f_s. \end{aligned} \tag{3.11}$$

Proof. We consider F with respect to the mixed tensor h_i^j. By equation (3.5) and (3.6) we have

$$\begin{aligned} F_t &= F^{ij}(h_i^j)_t = F^{ij}\left(u\tilde{h}_i^j + \nu^{n+1}\delta_{ij}\right)_t \\ &= uF^{ij}[(F-\sigma)u]_i^j + u^2(F-\sigma)F^{ij}\tilde{h}_i^k\tilde{h}_k^j \\ &+ uw(F-\sigma)F^{ij}\tilde{h}_i^j - [(F-\sigma)u]_i u^i \sum f_s \\ &= uF^{ij}[(F-\sigma)u]_i^j + (F-\sigma)\left[\sum f_s\kappa_s^2 - 2\nu^{n+1}F + (\nu^{n+1})^2\sum f_s\right] \\ &+ w(F-\sigma)(F - \nu^{n+1}\sum f_s) - [(F-\sigma)u]_i u^i \sum f_s. \end{aligned} \tag{3.12}$$

□

4. Preserving convexity

Let u be an admissible solution of (1.19) on the domain $\overline{\Omega}\times[0,T)$. In this section, we are going to prove that if the initial surface is convex and satisfying $f(\kappa[\Sigma_0]) > \sigma$, then during the evolution, the graph $\Sigma(t) = (x, u(x,t))$ stays convex and satisfies $f(\kappa[\Sigma(t)]) > \sigma$, for any $t \in [0,T)$. For convenient, from now on we always choose $\tau_1, \cdots, \tau_n$ to be orthonormal in hyperbolic metrics, i.e., $g_{ij} = \delta_{ij}$ and $\tilde{g}_{ij} = u^2\delta_{ij}$.

Lemma 4.1. *If the initial surface Σ_0 is convex and satisfies $f(\Sigma_0) > \sigma$, then for any $t \in [0,T)$, the flow surface $\Sigma(t)$ stays convex. In particular, $f(\Sigma(t)) > \sigma,\ \forall (x,t) \in \Omega \times (0,T)$.*

Proof. By assumption (1.5) we can see that the convexity preserving property follows directly from $f(\Sigma(t)) > \sigma,\ (x,t) \in \Omega\times(0,t)$. Therefore, in the following we only need to show that if $f(\Sigma_0) > \sigma$, then $f(\Sigma(t)) > \sigma,\ \forall (x,t) \in \Omega\times(0,t)$.

First, instead of showing strict inequality, we will show $f(\Sigma(t)) \geq \sigma,\ \forall (x,t) \in \Omega\times(0,t)$. Combining equation (2.6) and Lemma 3.5 we have

$$\begin{aligned} & \frac{\partial F}{\partial t} - F^{ij}\nabla_{ij}F \\ & = (F-\sigma)\left[\sum f_s\kappa_s^2 - \nu^{n+1}F + (\nu^{n+1})^2\sum f_s + wF - 2\sum f_s\right]. \end{aligned} \tag{4.1}$$

Consider function $\tilde{F} = e^{-\lambda t}(F-\sigma)$, where $\lambda > 0$ to be determined later. By equation (4.1) we know that $\tilde{F}$ satisfies

$$\begin{aligned} & \frac{\partial \tilde{F}}{\partial t} - F^{ij}\nabla_{ij}\tilde{F} \\ & = \tilde{F}\left[\sum f_s\kappa_s^2 - \nu^{n+1}F + (\nu^{n+1})^2\sum f_s + wF - 2\sum f_s - \lambda\right]. \end{aligned} \tag{4.2}$$

If $\tilde{F}$ achieves its negative minimum at an interior point $(x_0, t_0) \in \Omega_T = \Omega \times (0, T)$, then at this point we would have

$$0 \geq \tilde{F}\left[\sum f_s \kappa_s^2 - \nu^{n+1}F + (\nu^{n+1})^2 \sum f_s + wF - 2\sum f_s - \lambda\right].$$

Choosing $\lambda > \max_{\overline{\Omega}\times[0,T^*]} \left|f_s\kappa_s^2 - \nu^{n+1}F + (\nu^{n+1})^2 \sum f_s + wF - 2\sum f_s\right|$, where $0 < t_0 < T^* < T$, leads to a contradiction.

Now we are ready to show the strict inequality. Under the hypothesis $f(\Sigma_0) > \sigma$, assume $t_0 \in [0, T)$ is the first time such that $F(\Sigma(x_0, t_0)) = \sigma, (x_0, t_0) \in \Omega \times (0, T)$. Let $\tilde{F}^\epsilon = e^{-\lambda t}(F - \sigma) - \epsilon e^{-\lambda t}$, where $0 < \epsilon < \inf_{x\in\overline{\Omega}}\{f(\Sigma_0(x)) - \sigma\}$ small enough such that $\tilde{F}^\epsilon(\Sigma(x', t'))$ is a local minimum and $t_0 - t'$ is very small. Then at the point (x', t') we would have

$$0 \geq \frac{\partial \tilde{F}^\epsilon}{\partial t} - F^{ij}\nabla_{ij}\tilde{F}^\epsilon > e^{-\lambda t'}\epsilon\lambda\left(2e^{-\lambda(t_0 - t')} - 1\right) > 0,$$

which leads to a contradiction. □

Similarly we have

Corollary 4.2. *Let $\Sigma(t) = \{(x, u(x, t)), (x, t) \in \Omega \times [0, T)\}$ denote the flow surface, $f(\Sigma_0) > \sigma$, and u satisfies equation (1.12), then there exists a constant C only depends on u_0, such that*

$$F - \sigma \leq Ce^{\lambda(T^*)t} \quad \forall t \in [0, T^*),\ 0 < T^* < T. \tag{4.3}$$

Proof. We still consider the function $\tilde{F} = e^{-\lambda t}(F - \sigma)$ in $\Omega \times [0, T^*),\ 0 < T^* < T$ where λ chosen in the same way as before, then by Lemma 4.1 we have

$$\frac{\partial \tilde{F}}{\partial t} - F^{ij}\nabla_{ij}\tilde{F} < 0 \text{ in } \Omega \times [0, T^*).$$

Now we apply maximum principle and conclude that $\tilde{F}$ achieves its maximum at the parabolic boundary. By Theorem 3.1 we know that $F \equiv \sigma$ on $\partial\Omega \times (0, T)$, therefore let $C = \max_{x\in\overline{\Omega}} F(\Sigma_0(x)) - \sigma$, we get (4.3). □

Remark 4.3. From Corollary 4.2, we can see that for any fixed $0 < T^* < T$, there exists a constant C only depends on initial surface Σ_0 and T^*, such that for any $0 \leq t \leq T^*$, we have $F < C$.

5. Gradient estimates

In this section we shall show that for $t \in (0, T)$ an upward unit normal of the solution tends to a fixed asymptotic angle with our axis e_{n+1} on approaching to the boundary. Combining this with following results gives us a global gradient bound for the solution.

The following lemma is similar to Theorem 3.1 of [GS10].

Lemma 5.1. *Let $\Sigma(t) = \{(x, u(x,t)) : (x,t) \in \Omega_T\}$ be the flow surfaces with $u(x,t)$ is an admissible solution of equation (1.19). Then for $\epsilon > 0$ sufficiently small,*

$$\frac{\sigma - \nu^{n+1}}{u} < \frac{\sqrt{1-\sigma^2}}{r_1} + \frac{\epsilon(1+\sigma)}{r_1^2} \text{ on } \partial\Omega \times (0,T), \tag{5.1}$$

where r_1 is the maximal radius of exterior tangent sphere to $\partial\Omega$.

Proof. We first assume $r_1 < \infty$. Let Γ^ϵ denote the vertical ϵ-lift of boundary Γ, for a fixed point $x_0 \in \Gamma^\epsilon$, let $\mathbf{e_1}$ be the outward unit normal vector to Γ^ϵ at x_0. Let B_1 be a ball in $\mathbb{R}^{n+1}$ of radius R_1 centered at $a = (x_0 + r_1\mathbf{e}_1, R_1\sigma)$ where R_1 satisfies $R_1^2 = r_1^2 + (R_1\sigma - \epsilon)^2$.

Note that $B_1 \cap P(\epsilon) = \{x \in \mathbb{R}^{n+1} | x_{n+1} = \epsilon\}$ is an n-ball of radius r_1, which externally tangent to Γ^ϵ. By Lemma 3.3 of [LX10], we know that $B_1 \cap \Sigma(t) = \emptyset$, for any $t \in [0,T)$ hence at x_0, we have

$$\nu^{n+1} > -\frac{u - \sigma R_1}{R_1}.$$

By an easy computation we can get

$$R_1 \geq \frac{r_1^2}{\sqrt{(1-\sigma^2)r_1^2 + (1+\sigma)\epsilon}}$$

thus equation (5.1) is proved. If $r_1 = \infty$, then in the above argument one can replace r_1 by any $r > 0$ and then let $r \to \infty$. □

Proposition 5.2. *Let $\Sigma(t)$ be the flow surfaces, where $\Sigma(t) = \{(x, u(x,t)) : (x,t) \in \Omega_T\}$ and $u(x,t)$ satisfies the AMGCF equation (1.19). Then*

$$\frac{1}{\nu^{n+1}} \leq \max\left\{\frac{\max_{\Omega_T} u}{u}, \max_{\partial\Omega_T} \frac{1}{\nu^{n+1}}\right\}, \tag{5.2}$$

where $\Omega_T = \Omega \times [0,T)$.

Proof. Let $h = uw$ and suppose that h obtains its maximum at an interior point (x_0, t_0), then at this point we have

$$\partial_i h = (\delta_{ki} + u_k u_i + u u_{ki})\frac{u_k}{w} = 0, \text{ for } \forall\, 0 \leq i \leq n.$$

By Lemma 4.1 we know that $\Sigma(t_0)$ is strictly locally convex. According to Theorem 1.1, this implies that $\nabla u = 0$ at (x_0, t_0), thus the conclusion follows immediately. □

Now we can apply equation (2.5) and (2.6) to prove the following theorem.

Theorem 5.3. *Consider the flow surfaces $\Sigma(t)$, where $\Sigma(t)$ is supposed to be globally a graph:*

$$\Sigma(t) = \{(x, u(x,t)) : (x,t) \in \Omega_T\}$$

and $u(x,t)$ satisfies the AMGCF equation (1.19), then we have

$$\frac{\sigma - \nu^{n+1}}{u} \leq \max\left\{\frac{\sigma - \frac{1}{3}\sigma}{u}, \max_{\partial\Omega_T} \frac{\sigma - \nu^{n+1}}{u}\right\}. \tag{5.3}$$

Proof. By equation (2.3), (2.5) and let $\tilde{g}_{ij} = u^2\delta_{ij}$

$$\begin{aligned}\nabla_{ij}\frac{1}{u} &= -\frac{1}{u^2}\tilde{\nabla}_{ij}u + \frac{1}{u^3}\tilde{g}^{kl}u_ku_l\tilde{g}_{ij}\\ &= -\frac{1}{u^2}\tilde{h}_{ij}\nu^{n+1} + \frac{1}{u^3}\tilde{g}^{kl}u_ku_l\tilde{g}_{ij}\\ &= -\frac{\nu^{n+1}}{u}\left(h_{ij} - \frac{\nu^{n+1}}{u^2}\tilde{g}_{ij}\right) + \frac{1}{u^3}\tilde{g}^{kl}u_ku_l\tilde{g}_{ij}\end{aligned} \tag{5.4}$$

hence,

$$\begin{aligned}F^{ij}\nabla_{ij}\frac{1}{u} &= -\frac{\nu^{n+1}}{u}F + \frac{(\nu^{n+1})^2}{u^3}\sum F^{ij}\tilde{g}_{ij} + \frac{1}{u^3}u_ku^k\sum F^{ij}\tilde{g}_{ij}\\ &= -\frac{\nu^{n+1}}{u}F + \frac{(\nu^{n+1})^2}{u}\sum f_k + \frac{1-(\nu^{n+1})^2}{u}\sum f_k\\ &= -\frac{\nu^{n+1}}{u}F + \frac{1}{u}\sum f_k.\end{aligned} \tag{5.5}$$

Moreover,

$$\nabla_{ij}\frac{\nu^{n+1}}{u} = \nu^{n+1}\nabla_{ij}\frac{1}{u} + \frac{1}{u}\tilde{\nabla}_{ij}\nu^{n+1} - \frac{1}{u^2}\tilde{g}^{kl}u_k(\nu^{n+1})_l\tilde{g}_{ij}. \tag{5.6}$$

We recall the identities in $\mathbb{R}^{n+1}$

$$\left(\nu^{n+1}\right)_i = -\tilde{h}_{ij}\tilde{g}^{jk}u_k \tag{5.7}$$

$$\tilde{\nabla}_{ij}\nu^{n+1} = -\tilde{g}^{kl}\left(\nu^{n+1}\tilde{h}_{il}\tilde{h}_{kj} + u_l\tilde{\nabla}_k\tilde{h}_{ij}\right). \tag{5.8}$$

By equation (5.5), (5.6) and (5.8) we see that

$$\begin{aligned}&F^{ij}\nabla_{ij}\frac{\nu^{n+1}}{u}\\ &= \nu^{n+1}F^{ij}\nabla_{ij}\frac{1}{u} + \frac{1}{u}F^{ij}\tilde{\nabla}_{ij}\nu^{n+1} - \frac{1}{u^2}\tilde{g}^{kl}u_k(\nu^{n+1})_lF^{ij}\tilde{g}_{ij}\\ &= -\frac{(\nu^{n+1})^2}{u}F + \frac{\nu^{n+1}}{u}\sum f_k + \frac{1}{u}F^{ij}\left[-\tilde{g}^{kl}(\nu^{n+1}\tilde{h}_{il}\tilde{h}_{kj} + u_l\tilde{\nabla}_k\tilde{h}_{ij})\right]\\ &- \frac{1}{u^2}\tilde{g}^{kl}u_k(\nu^{n+1})_lF^{ij}\tilde{g}_{ij}.\end{aligned} \tag{5.9}$$

As a hypersurface in $\mathbb{R}^{n+1}$, it follows from equation (2.4) that for any $0 \le t < T$, $\Sigma(t)$ satisfies

$$f\left(u\tilde{\kappa}_1 + \nu^{n+1}, \cdots, u\tilde{\kappa}_n + \nu^{n+1}\right) = F \tag{5.10}$$

or equivalently,

$$F\left(\left\{u\tilde{g}^{sk}\tilde{h}_{kr} + \nu^{n+1}\delta_{sr}\right\}\right) = F. \tag{5.11}$$

Differentiating equation (5.11) and using $\tilde{g}^{sr} = \frac{\delta_{sr}}{u^2}$ we obtain

$$F_i = \frac{u_i}{u}F - \frac{u_i}{u}\nu^{n+1}\sum f_k + \frac{1}{u}F^{sr}\tilde{\nabla}_i\tilde{h}_{sr} + \left(\nu^{n+1}\right)_i\sum f_k. \tag{5.12}$$

Combining lemma 3.3 and equation (5.12) we derive

$$
\begin{aligned}
&\left(\frac{\nu^{n+1}}{u}\right)_t = \frac{\nu_t^{n+1}}{u} - \frac{\nu^{n+1}}{u^2}u_t \\
&= \frac{1}{u}\left\{-\tilde{g}^{ij}[(F-\sigma)u]_i u_j\right\} - \frac{\nu^{n+1}}{u^2}u_t \\
&= -\tilde{g}^{ij}F_i u_j - \frac{(F-\sigma)}{u}\tilde{g}^{ij}u_i u_j - \frac{(F-\sigma)}{u} \\
&= -u^i\left(\frac{u_i}{u}F - \frac{u_i}{u}\nu^{n+1}\sum f_k + \frac{1}{u}F^{st}\tilde{\nabla}_i\tilde{h}_{st} + (\nu^{n+1})_i\sum f_k\right) \\
&- \frac{(F-\sigma)}{u}\tilde{g}^{ij}u_i u_j - \frac{(F-\sigma)}{u} \\
&= -\frac{|\tilde{\nabla}u|^2}{u}F + \frac{|\tilde{\nabla}u|^2}{u}\nu^{n+1}\sum f_k - \frac{u^i}{u}F^{st}\tilde{\nabla}_i\tilde{h}_{st} - u^i(\nu^{n+1})_i\sum f_k \\
&- \frac{(F-\sigma)}{u}(|\tilde{\nabla}u|^2+1).
\end{aligned}
\tag{5.13}
$$

Finally we get

$$
\begin{aligned}
&\left(\frac{\partial}{\partial t} - F^{ij}\nabla_{ij}\right)\frac{\nu^{n+1}}{u} \\
&= -\frac{|\tilde{\nabla}u|^2}{u}F + \frac{|\tilde{\nabla}u|^2}{u}\nu^{n+1}\sum f_k - \frac{u^i}{u}F^{st}\tilde{\nabla}_i\tilde{h}_{st} \\
&- u^i(\nu^{n+1})_i\sum f_k - \frac{(F-\sigma)}{u}(|\tilde{\nabla}u|^2+1) + \frac{(\nu^{n+1})^2}{u}F - \frac{\nu^{n+1}}{u}\sum f_k \\
&- \frac{1}{u}F^{ij}\left[-\tilde{g}^{kl}(\nu^{n+1}\tilde{h}_{il}\tilde{h}_{kj} + u_l\tilde{\nabla}_k\tilde{h}_{ij})\right] + \frac{1}{u^2}\tilde{g}^{kl}u_k(\nu^{n+1})_l F^{ij}\tilde{g}_{ij} \\
&= -\frac{|\tilde{\nabla}u|^2}{u}F + \frac{\nu^{n+1} - (\nu^{n+1})^3}{u}\sum f_k - \frac{(F-\sigma)}{u}(|\tilde{\nabla}u|^2+1) \\
&+ \frac{(\nu^{n+1})^2}{u}F - \frac{\nu^{n+1}}{u}\sum f_k + \frac{1}{u}F^{ij}\tilde{g}^{kl}\nu^{n+1}\tilde{h}_{il}\tilde{h}_{kj} \\
&= -\frac{1}{u}F - \frac{(F-\sigma)}{u}(|\tilde{\nabla}u|^2+1) + \frac{\nu^{n+1}}{u}\sum f_k\kappa_k^2.
\end{aligned}
\tag{5.14}
$$

By a simple computation we have

$$
\left(\frac{\partial}{\partial t} - F^{ij}\nabla_{ij}\right)\frac{1}{u} = -\frac{(F-\sigma)}{u\nu^{n+1}} + \frac{\nu^{n+1}}{u}F - \frac{1}{u}\sum f_k. \tag{5.15}
$$

Therefore,

$$
\begin{aligned}
&\left(\frac{\partial}{\partial t}-F^{ij}\nabla_{ij}\right)\frac{\sigma-\nu^{n+1}}{u}\\
&=-\frac{\sigma(F-\sigma)}{u\nu^{n+1}}+\frac{\sigma\nu^{n+1}}{u}F-\frac{\sigma}{u}\sum f_k+\frac{1}{u}F\\
&+\frac{(F-\sigma)}{u}(2-(\nu^{n+1})^2)-\frac{\nu^{n+1}}{u}\sum f_k\kappa_k^2\\
&\leq\frac{1}{u}(F-\sigma\sum f_k)+\frac{(F-\sigma)}{u}\left(2-(\nu^{n+1})^2-\frac{\sigma}{\nu^{n+1}}\right)\\
&+\frac{\nu^{n+1}}{u}\left(\sigma F-\frac{F^2}{\sum f_k}\right)\\
&=\frac{1}{u}(F-\sigma\sum f_k)\left(1-\frac{\nu^{n+1}F}{\sum f_k}\right)+\frac{(F-\sigma)}{u}\left(2-(\nu^{n+1})^2-\frac{\sigma}{\nu^{n+1}}\right)
\end{aligned}
\tag{5.16}
$$

where we applied inequality $\sum f_k\kappa_k^2\geq\frac{F^2}{\sum f_k}$. If $\frac{\sigma-\nu^{n+1}}{u}$ achieves its maximum at an interior point (x_0,t_0), then at this point we have

$$
\begin{aligned}
0&\leq\frac{1}{u}(F-\sigma\sum f_k)\left(1-\frac{\nu^{n+1}F}{\sum f_k}\right)+\frac{(F-\sigma)}{u}\left(2-(\nu^{n+1})^2-\frac{\sigma}{\nu^{n+1}}\right)\\
&=\frac{F-\sigma}{u}\left(2-(\nu^{n+1})^2-\frac{\sigma}{\nu^{n+1}}+\frac{F-\sigma\sum f_k}{(F-\sigma)}-\frac{\nu^{n+1}F}{\sum f_k}\frac{(F-\sigma\sum f_k)}{(F-\sigma)}\right).
\end{aligned}
\tag{5.17}
$$

when $F\geq\sigma\sum f_k$

$$0\leq\frac{(F-\sigma)}{u}\left(3-\frac{\sigma}{\nu^{n+1}}\right),$$

when $F<\sigma\sum f_k$

$$0\leq\frac{(F-\sigma)}{u}\left(2-\frac{\sigma}{\nu^{n+1}}\right).$$

Thus by Lemma 4.1 we have when $\nu^{n+1}<\frac{\sigma}{3}$ at (x_0,t_0),

$$\left(\frac{\partial}{\partial t}-F^{ij}\nabla_{ij}\right)\frac{\sigma-\nu^{n+1}}{u}<0,$$

which leads to a contradiction.

Therefore we conclude that

$$\frac{\sigma-\nu^{n+1}}{u}\leq\max\left\{\frac{\sigma-\frac{1}{3}\sigma}{u},\max_{\partial\Omega_T}\frac{\sigma-\nu^{n+1}}{u}\right\}.$$

□

Combining Lemma 5.1, Proposition 5.2 and Theorem 5.3 gives

Corollary 5.4. *For any $\epsilon>0$ sufficiently small, any admissible solution u^ϵ of the Dirichlet problem (1.19) satisfies the a priori estimates*

$$|\nabla u^\epsilon|\leq C\ \text{in}\ \overline{\Omega}_T,\tag{5.18}$$

where C is independent of ϵ and T.

6. C^2 BOUNDARY ESTIMATES

In this section, we will establish boundary estimates for second order spatial derivatives of the admissible solutions to the Dirichlet problem (1.19). Following the notations in subsection 2.2 we can rewrite equation (1.19) as follows:

$$
\begin{cases} \frac{1}{uw}u_t - F\left(\frac{1}{w}(\delta_{ij} + u\gamma^{ik}u_{kl}\gamma^{lj})\right) = -\sigma & \text{on } \Omega_T, \\ u(x,0) = u_0^\epsilon & \text{on } \Omega \times \{0\}, \\ u(x,t) = \epsilon & \text{on } \partial\Omega \times [0,T). \end{cases} \tag{6.1}
$$

And from now on we denote

$$
G(D^2u, Du, u, u_t) = \frac{1}{uw}u_t - F. \tag{6.2}
$$

Theorem 6.1. *Suppose f satisfies equation (1.3)-(1.9). If ϵ is sufficiently small,*

$$
u|D^2u| \le C \text{ on } \partial\Omega \times [0,T), \tag{6.3}
$$

where C is independent of ϵ.

Remark 6.2. The following proof shows that C does not depend on ϵ, but depends on T. In section 7 we will show that in fact C is also independent of T.

Note that

$$
G^{kl} := \frac{\partial G}{\partial u_{kl}} = -\frac{u}{w}F^{ij}\gamma^{ik}\gamma^{lj}, \tag{6.4}
$$

$$
G^{kl}u_{kl} = -F + \frac{1}{w}\sum F^{ii}, \tag{6.5}
$$

$$
\begin{aligned} G_u := \frac{\partial G}{\partial u} &= -\frac{1}{wu^2}u_t - \frac{1}{w}F^{ij}\gamma^{ik}u_{kl}\gamma^{lj} \\ &= -\frac{(F-\sigma)}{u} - F^{ij}\left(\frac{a_{ij}}{u} - \frac{1}{uw}\delta_{ij}\right) \\ &= -\frac{2F}{u} + \frac{\sigma}{u} + \frac{1}{wu}\sum F^{ii}, \end{aligned} \tag{6.6}
$$

$$
G^t := \frac{\partial G}{\partial u_t} = \frac{1}{uw}, \tag{6.7}
$$

$$
\begin{aligned} G^s &:= \frac{\partial G}{\partial u_s} \\ &= -\frac{u_t u_s}{uw^3} + \frac{u_s}{w^2}F + \frac{2}{w}F^{ij}a_{ik}\left(\frac{wu_k\gamma^{sj} + u_j\gamma^{ks}}{1+w}\right) - \frac{2}{w^2}F^{ij}u_i\gamma^{sj} \\ &= -\frac{(F-\sigma)}{w^2}u_s + \frac{u_s}{w^2}F + \frac{2}{w}F^{ij}a_{ik}\left(\frac{wu_k\gamma^{sj} + u_j\gamma^{ks}}{1+w}\right) - \frac{2}{w^2}F^{ij}u_i\gamma^{sj} \\ &= \frac{u_s}{w^2}\sigma + \frac{2}{w}F^{ij}a_{ik}\left(\frac{wu_k\gamma^{sj} + u_j\gamma^{ks}}{1+w}\right) - \frac{2}{w^2}F^{ij}u_i\gamma^{sj}. \end{aligned} \tag{6.8}
$$

Thus

$$G^s u_s = \frac{w^2-1}{w^2}\sigma + \frac{2}{w^2}F^{ij}a_{ik}u_k u_j - \frac{2}{w^3}F^{ij}u_i u_j. \tag{6.9}$$

And similar to equation (5.4) in [GS08] we have

$$\sum |G^s| \le C(\sum F^{ii} + F). \tag{6.10}$$

Next, we consider the partial linearized operator of G at u:

$$L = G^t\partial_t + G^{kl}\partial_k\partial_l + G^s\partial_s.$$

By equation (6.5),(6.7) and (6.9) we get

$$\begin{aligned} Lu &= \frac{1}{uw}u_t - F + \frac{1}{w}\sum F^{ii} + \frac{w^2-1}{w^2}\sigma + \frac{2}{w^2}F^{ij}a_{ik}u_k u_j - \frac{2}{w^3}F^{ij}u_i u_j \\ &= -\frac{1}{w^2}\sigma + \frac{1}{w}\sum F^{ii} + \frac{2}{w^2}F^{ij}a_{ik}u_k u_j - \frac{2}{w^3}F^{ij}u_i u_j, \end{aligned} \tag{6.11}$$

hence

$$\begin{aligned} &L\left(\frac{1}{u}\right) = -\frac{1}{u^2}Lu + \frac{2}{u^3}G^{kl}u_k u_l \\ &= \frac{1}{u^2w^2}\sigma - \frac{1}{u^2w}\sum F^{ii} - \frac{2}{u^2w^2}F^{ij}a_{ik}u_k u_j \\ &+ \frac{2}{w^3u^2}F^{ij}u_i u_j - \frac{2}{u^2w}F^{ij}\gamma^{is}u_s\gamma^{rj}u_r \\ &= \frac{1}{w^2u^2}\sigma - \frac{1}{wu^2}\sum F^{ii} - \frac{2}{w^2u^2}F^{ij}a_{ik}u_k u_j. \end{aligned} \tag{6.12}$$

Lemma 6.3. *Suppose that f satisfies equation (1.3), (1.4), (1.7) and (1.8). Then*

$$L\left(1-\frac{\epsilon}{u}\right) \ge \frac{\epsilon(1-\sigma)}{wu^2}\sum F^{ii} \text{ in } \Omega_T. \tag{6.13}$$

Proof. Since $\{F^{ij}\}$ and $\{a_{ij}\}$ are both positive definite and can be diagonalized simultaneously, we see that

$$F^{ij}a_{ik}\xi_k\xi_j \ge 0, \ \forall \xi \in \mathbb{R}^n. \tag{6.14}$$

Combining with equation (6.12)

$$\begin{aligned} &L\left(1-\frac{\epsilon}{u}\right) = -\epsilon L\left(\frac{1}{u}\right) \\ &= \frac{-\epsilon}{u^2w^2}\sigma + \frac{\epsilon}{u^2w}\sum F^{ii} + \frac{2\epsilon}{w^2u^2}F^{ij}a_{ik}u_k u_j \\ &\ge \frac{\epsilon\left(1-\frac{\sigma}{w}\right)}{wu^2}\sum F^{ii} \ge \frac{\epsilon(1-\sigma)}{u^2w}\sum F^{ii}. \end{aligned} \tag{6.15}$$

□

Now we denote $\mathfrak{L} = G^t\partial t + G^{kl}\partial_k\partial_l + G^s\partial_s + G_u$, similar to [CNS84] we have

Lemma 6.4. *Suppose that f satisfies equation (1.3), (1.4), (1.7) and (1.8). Then*

$$\mathfrak{L}(x_i u_j - x_j u_i) = 0, \ \mathfrak{L}(u_i) = 0, \ 1 \le i, j \le n. \tag{6.16}$$

Proof of Theorem 6.1. Consider an arbitrary point on $\partial\Omega$, which we may assume to be the origin of $\mathbb{R}^n$ and choose the coordinates so that the positive x_n axis is the interior normal to $\partial\Omega$ at the origin. There exists a uniform constant $r > 0$ such that $\partial\Omega \cap B_r(0)$ can be represented as a graph

$$x_n = \rho(x') = \frac{1}{2}\sum_{\alpha,\beta<n} B_{\alpha\beta}x_\alpha x_\beta + O(|x'|^3),\ x' = (x_1, \cdots, x_{n-1}).$$

Since $u \equiv \epsilon$, on $\partial\Omega \times [0, T)$, i.e., $u(x', \rho(x')) \equiv \epsilon$ for $\forall t \in [0, T)$, at the origin we have

$$u_\alpha + u_n B_{\alpha\beta}x_\beta = 0,\ u_{\alpha\beta} + u_n\rho_{\alpha\beta} = 0,\ \forall t \in [0, T) \text{ and } \alpha, \beta < n.$$

Consequently,

$$|u_{\alpha\beta}(0, t)| \leq C|Du(0, t)|,\ \ \forall t \in [0, T) \text{ and } \alpha, \beta < n, \tag{6.17}$$

where C depends only on the maximal (Euclidean principal) curvature of $\partial\Omega$. Following [CNS84] let $T_\alpha = \partial_\alpha + \sum_{\beta<n} B_{\alpha\beta}(x_\beta\partial_n - x_n\partial_\beta)$, then for fixed $\alpha < n$, we have

$$|T_\alpha u| \leq C_1|x|^2,\ \text{on } \{\partial\Omega \cap B_\epsilon(0)\} \times [0, T), \tag{6.18}$$

$$|T_\alpha u| \leq C_1,\ \text{in } \{\Omega \cap B_\epsilon(0)\} \times [0, T), \tag{6.19}$$

where C_1 is independent of ϵ and T. Moreover by Lemma 6.4

$$\mathfrak{L}T_\alpha u = 0. \tag{6.20}$$

Therefore

$$\begin{aligned} |L(T_\alpha u)| &= |\mathfrak{L}(T_\alpha u) - G_u T_\alpha u| \\ &= |G_u T_\alpha u| \leq C_1|G_u| \\ &\leq \frac{C_2}{u}(\sum F^{ii} + F) \\ &\leq \frac{C_2}{u}\sum F^{ii}\ \ \text{in } \{\Omega \cap B_\epsilon(0)\} \times [0, T). \end{aligned} \tag{6.21}$$

Note that the last inequality comes from equation (1.11), Corollary 4.2 and Remark 4.3. Hence C_2 is some constant only depending on T. By equation (6.4), (6.10) and Lemma 2.1 in [GS08]

$$\begin{aligned} |L(|x|^2)| &= \left|G^{kl}\partial_k\partial_l(|x|^2) + G^s\partial_s(|x|^2)\right| \\ &= |2\sum G^{kk} + 2\sum x_s G^s| \\ &\leq C_3(u\sum F^{ii} + \epsilon|G_s|) \leq C_3 u\sum F^{ii}\ \ \text{in } \{\Omega \cap B_\epsilon(0)\} \times [0, T), \end{aligned} \tag{6.22}$$

for the same reason as before we know that C_3 only depends on T as well.

Now consider function

$$\Phi = A\left(1 - \frac{\epsilon}{u}\right) + B|x|^2 \pm T_\alpha u.$$

First choose $B \geq \frac{C_1}{\epsilon^2}$, then we have $\Phi \geq 0$ on $\{\partial(\Omega \cap B_\epsilon(0))\} \times [0, T)$.

Next consider Φ on $(\Omega\cap B_\delta(0))\times\{0\}$, where $\delta>\epsilon>0$ is small enough. By using Taylor's theorem we have

$$\begin{aligned}\Phi &= A\left(1-\frac{\epsilon}{u_0}\right)+B|x|^2\pm T_\alpha u_0\\ &\geq A\left(1-\frac{\epsilon}{\epsilon+a_1x_n}\right)+B|x|^2-b_1x_n-b_2|x|^2\\ &\geq \left(\frac{Aa_1}{1+a_1}-b_1\right)x_n+(B-b_2)\,|x|^2,\end{aligned}$$

where $u_0\geq\epsilon+a_1x_n$, $|T_\alpha u_0|\leq b_1x_n+b_2|x|^2$ in $\Omega\cap B_\delta(0)$ and $a_1,b_1,b_2>0$. (The reason of the existence of a_1 can be found in section 3 of [LX10] while the existence of b_i, $i=1,2$ is trivial.) Hence we conclude that when $A\geq\frac{b_1(1+a_1)}{a_1}$ and $B\geq\max\{\frac{C_1}{\epsilon^2},b_2\}$, $\Phi\geq0$ on $\{\partial(\Omega\cap B_\epsilon(0))\times[0,T)\}\cap\{(\Omega\cap B_\epsilon(0))\times\{0\}\}$.

Moreover, by (1.11),(6.21),(6.21) and Lemma 6.3

$$\begin{aligned}L(\Phi)&=AL\left(1-\frac{\epsilon}{u}\right)+BL(|x|^2)\pm L(T_\alpha u)\\ &\geq\frac{A\epsilon(1-\sigma)}{u^2w}-C_3Bu-\frac{C_2}{u}.\end{aligned}\tag{6.23}$$

Choosing $A\gg\frac{C_1C_3+C_2}{1-\sigma}$ such that $L\Phi\geq0$ in $\{\Omega\cap B_\epsilon\}\times[0,T)$, which implies that $\Phi\geq0$ in $\{\Omega\cap B_\epsilon\}\times[0,T)$. Since $\Phi(0,t)=0$, we have $\Phi_n(0,t)\geq0$, for any fixed $t\in[0,T)$. Thus

$$A\left(\frac{\epsilon}{u^2}u_n\right)\pm(T_\alpha u)_n\geq0\tag{6.24}$$

which implies, for any fixed $t\in[0,T)$,

$$|u_{\alpha n}(0,t)|\leq\frac{Au_n(0,t)}{u(0,t)}.\tag{6.25}$$

Since when $t=0$, $u_{nn}(0,0)$ is given we only care about the case when $t>0$. By Theorem 3.1, we know that $F\equiv\sigma$, on $\partial\Omega\times(0,T)$. Therefore we can establish $|u_{nn}(0,t)|$, $\forall t\in(0,T)$ in the same way as [GSZ09]. For completeness we include the argument here.

For a fixed $t\in(0,T)$, we may assume $(u_{\alpha\beta}(0,t))_{1\leq\alpha,\beta<n}$ to be diagonal. Then at the point $(0,t)$

$$A[u]=\frac{1}{w}\begin{bmatrix}1+uu_{11}&0&\cdots&\frac{uu_{1n}}{w}\\ 0&1+uu_{22}&\cdots&\frac{uu_{2n}}{w}\\ \vdots&\vdots&\ddots&\vdots\\ \frac{uu_{n1}}{w}&\frac{uu_{n2}}{w}&\cdots&1+\frac{uu_{nn}}{w^2}\end{bmatrix}$$

By lemma 1.2 in [CNS85], if $\epsilon u_{nn}(0,t)$ is very large, the eigenvalues $\lambda_1,\cdots,\lambda_n$ of $A[u]$ are given by

$$\begin{aligned}\lambda_\alpha&=\frac{1}{w}(1+\epsilon u_{\alpha\alpha}(0,t))+o(1),\ \alpha<n\\ \lambda_n&=\frac{\epsilon u_{nn}(0,t)}{w^3}\left(1+O\left(\frac{1}{\epsilon u_{nn}(0,t)}\right)\right).\end{aligned}\tag{6.26}$$

If $\epsilon u_{nn} \geq R$ where R is a uniform constant, then by (1.8), (1.9) and Lemma 5.1 we have

$$\sigma = \frac{1}{w} F(wA[u])(0,t) \geq (\sigma - C\epsilon)\left(1 + \frac{\epsilon_0}{2}\right) > \sigma$$

which is a contradiction. Therefore

$$|u_{nn}(0,t)| \leq \frac{R}{\epsilon}$$

and the proof is completed. □

7. C^2 GLOBAL ESTIMATES

Let $\Sigma(t) = \{(x, u(x,t)) \mid x \in \Omega, t \in [0,T)\}$ be the flow surfaces in $\mathbb{H}^{n+1}$ where $u(x,t)$ satisfies $u_t = uw(F - \sigma)$. For a fixed point $\mathbf{x}_0 \in \Sigma(t_0)$, $0 < t_0 < T$ we choose a local orthonormal frame $\tau_1, \cdots, \tau_n$ around $\mathbf{x}_0$ such that $h_{ij}(\mathbf{x}_0) = \kappa_i \delta_{ij}$, where $\kappa_1, \cdots, \kappa_n$ are the hyperbolic principal curvatures of $\Sigma(t_0)$ at $\mathbf{x}_0$. The calculations below are done at $\mathbf{x}_0$. In this section, for convenience we shall write $v_{ij} = \nabla_{ij} v$, $h_{ijk} = \nabla_k h_{ij}$, $h_{ijkl} = \nabla_{lk} h_{ij}$, etc.

Theorem 7.1. *Let $\Sigma(t) = \{(x, u(x,t)) \mid x \in \Omega, t \in [0,T)\}$ be the flow surfaces in $\mathbb{H}^{n+1}$ where $u(x,t)$ satisfies AMGCF equation (1.19) and*

$$\nu^{n+1} \geq 2a > 0 \ \text{ on } \Sigma(t),\ \forall t \in [0,T).$$

For $\mathbf{x} \in \Sigma(t)$, let $\kappa_{\max}(\mathbf{x})$ be the largest principal curvature of $\Sigma(t)$ at $\mathbf{x}$. Then

$$\max_{\overline{\Omega}_T} \frac{\kappa_{\max}}{\nu^{n+1} - a} \leq \max\left\{\frac{4}{a^3}, \max_{\partial\Omega_T} \frac{\kappa_{\max}}{\nu^{n+1} - a}\right\}, \tag{7.1}$$

where $\Omega_T = \Omega \times [0,T)$.

Since the proof of this Theorem is very complicated, we shall divide it into several parts.

To begin with, we denote

$$M_0 = \max_{\overline{\Omega}_T} \frac{\kappa_{\max}(x)}{\nu^{n+1} - a}. \tag{7.2}$$

Without loss of generality we may assume $M_0 > 0$ is attained at an interior point $\mathbf{x}_0 \in \Sigma(t_0)$, $t_0 \in (0,T)$. We may also assume $\kappa_1 = \kappa_{\max}(\mathbf{x}_0)$. Thus we say at $\mathbf{x}_0$, $\frac{h_{11}}{\nu^{n+1}-a}$ achieves its local maximum. Hence,

$$\frac{h_{11i}}{h_{11}} - \frac{\nabla_i \nu^{n+1}}{\nu^{n+1} - a} = 0, \tag{7.3}$$

$$\frac{h_{11ii}}{h_{11}} - \frac{\nabla_{ii} \nu^{n+1}}{\nu^{n+1} - a} \leq 0. \tag{7.4}$$

Lemma 7.2. *At $\mathbf{x}_0 \in \Sigma(t_0)$, $t_0 \in (0,T)$,*

$$\begin{aligned} \frac{\partial}{\partial t} h_{11} = {} & \nabla_{11} F - (F - \sigma)\kappa_1^2 + \kappa_1 \nu^{n+1}(F - \sigma) \\ & - \frac{\kappa_1}{\nu^{n+1}}(F - \sigma) + (F - \sigma)(\nu^{n+1})^2 - 2(F - \sigma). \end{aligned} \tag{7.5}$$

Proof. By Lemma 3.4 equation (3.8) and $\tilde{g}_{ij} = u^2\delta_{ij}$ we have,

$$\begin{aligned}\frac{\partial}{\partial t}h_{11} &= \frac{1}{u}\{\tilde{\nabla}_{11}[(F-\sigma)u] - u(F-\sigma)\tilde{h}_1^k\tilde{h}_{k1}\} - \frac{\tilde{h}_{11}}{u}w(F-\sigma)\\ &- [u(F-\sigma)]_k u^k - \frac{2(F-\sigma)\nu^{n+1}}{u}\tilde{h}_{11} - 2(F-\sigma).\end{aligned}\tag{7.6}$$

Recall equation (2.6) we get

$$\begin{aligned}&\tilde{\nabla}_{11}[(F-\sigma)u] = \nabla_{11}[(F-\sigma)u] - \frac{1}{u}\left\{2u_1[(F-\sigma)u]_1 - u^k[(F-\sigma)u]_k u^2\right\}\\ &= u\nabla_{11}F + (F-\sigma)\nabla_{11}u + 2F_1u_1 - \frac{2}{u}\left\{uu_1F_1 + u_1^2(F-\sigma)\right\} + uu^k[(F-\sigma)u]_k\\ &= u\nabla_{11}F + (F-\sigma)\nabla_{11}u - \frac{2u_1^2(F-\sigma)}{u} + u^k[(F-\sigma)u]_k u,\end{aligned}$$

inserting this into (7.6)

$$\begin{aligned}\frac{\partial}{\partial t}h_{11} &= \frac{1}{u}\left\{u\nabla_{11}F + (F-\sigma)\nabla_{11}u - \frac{2u_1^2(F-\sigma)}{u} + u^k[(F-\sigma)u]_k u\right\}\\ &- (F-\sigma)\tilde{h}_1^k\tilde{h}_{k1} - \frac{\tilde{h}_{11}}{u}w(F-\sigma) - [u(F-\sigma)]_k u^k\\ &- \frac{2(F-\sigma)\nu^{n+1}}{u}\tilde{h}_{11} - 2(F-\sigma)\\ &= \nabla_{11}F + \frac{(F-\sigma)}{u}\nabla_{11}u - \frac{2u_1^2(F-\sigma)}{u^2} - (F-\sigma)\tilde{h}_1^k\tilde{h}_{k1}\\ &- \frac{\tilde{h}_{11}}{u}w(F-\sigma) - \frac{2(F-\sigma)\nu^{n+1}}{u}\tilde{h}_{11} - 2(F-\sigma).\end{aligned}\tag{7.7}$$

Note that,

$$\nabla_{11}u = \tilde{\nabla}_{11}u + \frac{2u_1^2}{u} - u|\tilde{\nabla}u|^2,$$

$$\frac{\tilde{h}_{11}}{u} = h_{11} - \nu^{n+1},$$

$$\tilde{h}_1^k\tilde{h}_{k1} = \frac{1}{u^2}\tilde{h}_{1k}^2 = \frac{1}{u^2}(uh_{1k} - u\nu^{n+1}\delta_{1k})^2 = (h_{11} - \nu^{n+1})^2.$$

So we have,

$$
\begin{aligned}
\frac{\partial}{\partial t}h_{11} &= \nabla_{11}F + \frac{(F-\sigma)}{u}\left(\tilde{h}_{11}\nu^{n+1} + \frac{2u_1^2}{u} - u|\tilde{\nabla}u|^2\right) \\
&- \frac{2u_1^2}{u^2}(F-\sigma) - (F-\sigma)(h_{11}-\nu^{n+1})^2 - (h_{11}-\nu^{n+1})w(F-\sigma) \\
&- 2(F-\sigma)\nu^{n+1}(h_{11}-\nu^{n+1}) - 2(F-\sigma) \\
&= \nabla_{11}F + (F-\sigma)\nu^{n+1}(h_{11}-\nu^{n+1}) - (F-\sigma)(1-(\nu^{n+1})^2) \\
&- (F-\sigma)(h_{11}^2 - 2h_{11}\nu^{n+1} + (\nu^{n+1})^2) - (h_{11}-\nu^{n+1})w(F-\sigma) \\
&- 2(F-\sigma)\nu^{n+1}(h_{11}-\nu^{n+1}) - 2(F-\sigma) \\
&= \nabla_{11}F - (F-\sigma) - (F-\sigma)\kappa_1^2 + 2\kappa_1\nu^{n+1}(F-\sigma) \\
&- \frac{\kappa_1}{\nu^{n+1}}(F-\sigma) + (F-\sigma) - (F-\sigma)\nu^{n+1}(\kappa_1-\nu^{n+1}) - 2(F-\sigma) \\
&= \nabla_{11}F - (F-\sigma)\kappa_1^2 + \kappa_1\nu^{n+1}(F-\sigma) - \frac{\kappa_1}{\nu^{n+1}}(F-\sigma) \\
&+ (F-\sigma)(\nu^{n+1})^2 - 2(F-\sigma).
\end{aligned} \tag{7.8}
$$

□

proof of Theorem 7.1. Now we denote $\varphi = \frac{h_{11}}{\nu^{n+1}-a}$, where $\nu^{n+1} \geq 2a > 0$ on $\overline{\Omega}_T$. Then at $\mathbf{x}_0 \in \Sigma(t_0)$, we have

$$
\nabla_i\varphi = \frac{h_{11i}}{\nu^{n+1}-a} - \frac{h_{11}\nu_i^{n+1}}{(\nu^{n+1}-a)^2} = 0 \tag{7.9}
$$

$$
\nabla_{ii}\varphi = \frac{h_{11ii}}{\nu^{n+1}-a} - \frac{h_{11}\nabla_{ii}\nu^{n+1}}{(\nu^{n+1}-a)^2} \leq 0. \tag{7.10}
$$

Using Lemma 7.2 and equation (3.5) in Lemma 3.3 we get

$$
\begin{aligned}
\frac{\partial}{\partial t}\varphi &= \frac{\dot{h}_{11}}{\nu^{n+1}-a} - \frac{h_{11}\dot{\nu}^{n+1}}{(\nu^{n+1}-a)^2} \\
&= \frac{1}{\nu^{n+1}-a}\Big\{F^{ii}h_{ii11} + F^{ij,rs}h_{ij1}h_{rs1} - (F-\sigma)\kappa_1^2 \\
&+\kappa_1\nu^{n+1}(F-\sigma) - \frac{\kappa_1}{\nu^{n+1}}(F-\sigma) + (F-\sigma)(\nu^{n+1})^2 - 2(F-\sigma)\Big\} \\
&+ \frac{h_{11}}{(\nu^{n+1}-a)^2}u^k[(F-\sigma)u]_k.
\end{aligned} \tag{7.11}
$$

By equation (2.6) and (5.8)

$$
\begin{aligned}
\nabla_{ii}\nu^{n+1} &= \tilde{\nabla}_{ii}\nu^{n+1} + \frac{1}{u}\left(2u_i\nu_i^{n+1} - u^k\nu_k^{n+1}\tilde{g}_{ii}\right) \\
&= -\tilde{g}^{kl}\left(\nu^{n+1}\tilde{h}_{il}\tilde{h}_{ki} + u_l\tilde{\nabla}_k\tilde{h}_{ii}\right) + \frac{2}{u}u_i\nu_i^{n+1} - uu^k\nu_k^{n+1}\delta_{ii},
\end{aligned}
$$

we obtain

$$\begin{aligned}
&F^{ii}\nabla_{ii}\nu^{n+1} = -\nu^{n+1}F^{ii}\tilde{g}^{kl}\tilde{h}_{il}\tilde{h}_{ki} - F^{ii}u^k\tilde{\nabla}_k\tilde{h}_{ii}\\
&+\frac{2}{u}F^{ii}u_i\nu_i^{n+1} - uu^k\nu_k^{n+1}\sum f_i\\
&= -\nu^{n+1}\left(\sum f_i\kappa_i^2 - 2\nu^{n+1}F + (\nu^{n+1})^2\sum f_i\right) - F^{ii}u^k\tilde{\nabla}_k\tilde{h}_{ii}\\
&+\frac{2}{u}F^{ii}u_i\nu_i^{n+1} - uu^k\nu_k^{n+1}\sum f_i.
\end{aligned}$$

What's more, by the Codazzi and Gauss equations we have

$$h_{ii11} - h_{11ii} = (\kappa_i\kappa_1 - 1)(\kappa_i - \kappa_1) = \kappa_i^2\kappa_1 - \kappa_i\kappa_1^2 - \kappa_i + \kappa_1,$$

multiplying by F^{ii} and sum over i,

$$\sum F^{ii}(h_{ii11} - h_{11ii}) = \kappa_1\sum f_i\kappa_i^2 - \kappa_1^2F - F + \kappa_1\sum f_i. \tag{7.12}$$

Finally we get

$$\begin{aligned}
&\frac{\partial}{\partial t}\varphi - F^{ii}\nabla_{ii}\varphi = \frac{1}{\nu^{n+1}-a}\Big\{\kappa_1\sum f_i\kappa_i^2 - \kappa_1^2F - F\\
&+\kappa_1\sum f_i + F^{ij,rs}h_{ij1}h_{rs1} - (F-\sigma)\kappa_1^2 + \kappa_1\nu^{n+1}(F-\sigma)\\
&-\frac{\kappa_1}{\nu^{n+1}}(F-\sigma) + (F-\sigma)(\nu^{n+1})^2 - 2(F-\sigma)\Big\}\\
&+\frac{\kappa_1}{(\nu^{n+1}-a)^2}\Big\{uF_ku^k + |\tilde{\nabla}u|^2(F-\sigma) + F^{ii}\nabla_{ii}\nu^{n+1}\Big\}\\
&= \frac{1}{\nu^{n+1}-a}\Big\{\kappa_1\sum f_i\kappa_i^2 - \kappa_1^2F - F + \kappa_1\sum f_i\\
&+F^{ij,rs}h_{ij1}h_{rs1} - (F-\sigma)\kappa_1^2 + \kappa_1\nu^{n+1}(F-\sigma)\\
&-\frac{\kappa_1}{\nu^{n+1}}(F-\sigma) + (F-\sigma)(\nu^{n+1})^2 - 2(F-\sigma)\Big\}\\
&+\frac{\kappa_1}{(\nu^{n+1}-a)^2}\Big\{u\left(\frac{|\tilde{\nabla}u|^2}{u}F - \frac{|\tilde{\nabla}u|^2}{u}\nu^{n+1}\sum f_i\right)\\
&+|\tilde{\nabla}u|^2(F-\sigma) - \nu^{n+1}\left[\sum f_i\kappa_i^2 - 2\nu^{n+1}F + (\nu^{n+1})^2\sum f_i\right]\\
&+\frac{2}{u}\sum F^{ii}u_i\nu_i^{n+1}\Big\},
\end{aligned} \tag{7.13}$$

where we have used equation (5.12). Hence at $\mathbf{x}_0 \in \Sigma(t_0)$ we have

$$\begin{aligned}
&0 \le \kappa_1 f_i\kappa_i^2 - \kappa_1^2F - F + \kappa_1\sum f_i + F^{ij,rs}h_{ij1}h_{rs1}\\
&-(F-\sigma)\kappa_1^2 + \kappa_1\nu^{n+1}(F-\sigma) - \frac{\kappa_1}{\nu^{n+1}}(F-\sigma)\\
&+(F-\sigma)(\nu^{n+1})^2 - 2(F-\sigma) + \frac{\kappa_1}{\nu^{n+1}-a}\Big\{|\tilde{\nabla}u|^2F\\
&-|\tilde{\nabla}u|^2\nu^{n+1}\sum f_i + |\tilde{\nabla}u|^2(F-\sigma) - \nu^{n+1}\Big[\sum f_i\kappa_i^2 - 2\nu^{n+1}F\\
&+(\nu^{n+1})^2\sum f_i\Big] + \frac{2}{u}\sum F^{ii}u_i\nu_i^{n+1}\Big\},
\end{aligned} \tag{7.14}$$

which implies

$$
\begin{aligned}
0 \le& \left(-1-\kappa_1^2+\kappa_1\frac{1+(\nu^{n+1})^2}{\nu^{n+1}-a}\right)F+F^{ij,rs}h_{ij1}h_{rs1}\\
&+\left(\kappa_1-\frac{\kappa_1\nu^{n+1}}{\nu^{n+1}-a}\right)\left(\sum f_i+\sum f_i\kappa_i^2\right)+\frac{2\kappa_1}{\nu^{n+1}-a}\sum f_i\frac{u_i^2}{u^2}(\nu^{n+1}-\kappa_i)\\
&+(F-\sigma)\kappa_1\left(-\kappa_1+\nu^{n+1}-\frac{1}{\nu^{n+1}}+\frac{1-(\nu^{n+1})^2}{\nu^{n+1}-a}\right)-(F-\sigma).
\end{aligned}
\tag{7.15}
$$

Next we use an inequality due to Andrews [A94] and Gerhardt [G96] which states

$$
-F^{ij,kl}h_{ij1}h_{kl1}\ge 2\sum_{i\ge 2}\frac{f_i-f_1}{\kappa_1-\kappa_i}h_{i11}^2. \tag{7.16}
$$

Meanwhile at $\mathbf{x}_0\in\Sigma(t_0)$, we obtain from equation (5.7) and (7.9)

$$
h_{11i}=\frac{\kappa_1}{\nu^{n+1}-a}\frac{u_i}{u}(\nu^{n+1}-\kappa_i). \tag{7.17}
$$

Inserting into (7.16) we derive

$$
F^{ij,rs}h_{ij1}h_{rs1}\le 2\left(\frac{\kappa_1}{\nu^{n+1}-a}\right)^2\sum_{i\ge 2}\frac{f_1-f_i}{\kappa_1-\kappa_i}\frac{u_i^2}{u^2}(k_i-\nu^{n+1})^2. \tag{7.18}
$$

Moreover we may write

$$
\sum f_i+\sum f_i\kappa_i^2=(1-(\nu^{n+1})^2)\sum f_i+\sum(\kappa_i-\nu^{n+1})^2 f_i+2F\nu^{n+1}. \tag{7.19}
$$

Combining equation (7.15), (7.18) and (7.19) gives

$$
\begin{aligned}
0 \le& \left(-1-\kappa_1^2+\frac{1+(\nu^{n+1})^2}{\nu^{n+1}-a}\kappa_1\right)F+\frac{2\kappa_1}{\nu^{n+1}-a}\sum f_i\frac{u_i^2}{u^2}(\nu^{n+1}-\kappa_i)\\
&-\frac{a\kappa_1}{\nu^{n+1}-a}\left((1-(\nu^{n+1})^2)\sum f_i+\sum(\kappa_i-\nu^{n+1})^2 f_i+2F\nu^{n+1}\right)\\
&-2\left(\frac{\kappa_1}{\nu^{n+1}-a}\right)^2\sum_{i\ge 2}\frac{f_i-f_1}{\kappa_1-\kappa_i}\frac{u_i^2}{u^2}(\kappa_i-\nu^{n+1})^2\\
&+(F-\sigma)\kappa_1\left(-\kappa_1+\nu^{n+1}-\frac{1}{\nu^{n+1}}+\frac{1-(\nu^{n+1})^2}{\nu^{n+1}-a}\right)-(F-\sigma).
\end{aligned}
\tag{7.20}
$$

Note that (assuming $\kappa_1\ge\frac{2}{a}$) all terms on the right hand side are negative except possibly the ones in the sum involving $(\nu^{n+1}-\kappa_i)$ and only if $\kappa_i<\nu^{n+1}$.

Therefore define

$$
I=\left\{i:\kappa_i-\nu^{n+1}\le-\theta\kappa_1\right\},
$$

$$
J=\left\{i:-\theta\kappa_1<\kappa_i-\nu^{n+1}<0,\ f_i<\theta^{-1}f_1\right\},
$$

$$
L=\left\{i:-\theta\kappa_1<\kappa_i-\nu^{n+1}<0,\ f_i\ge\theta^{-1}f_1\right\},
$$

where $\theta \in (0,1)$ is to be chosen later. We get

$$\begin{aligned}
&\frac{-1}{\nu^{n+1}-a}\sum_{i\in I}(\kappa_i-\nu^{n+1})^2 f_i\\
&\leq \frac{\theta\kappa_1}{\nu^{n+1}-a}\sum_{i\in I}(\kappa_i-\nu^{n+1})f_i\\
&\leq \frac{\theta\kappa_1}{\nu^{n+1}-a}\sum_{i\in I}(\kappa_i-\nu^{n+1})f_i\frac{u_i^2}{u^2},
\end{aligned} \tag{7.21}$$

$$\sum_{i\in J}(\nu^{n+1}-\kappa_i)f_i\frac{u_i^2}{u^2}\leq \kappa_1 f_1. \tag{7.22}$$

Finally

$$\begin{aligned}
&\frac{-2\kappa_1^2}{(\nu^{n+1}-a)^2}\sum_{i\in L}\frac{f_i-f_1}{\kappa_1-\kappa_i}\frac{u_i^2}{u^2}(\kappa_i-\nu^{n+1})^2\\
&\leq \frac{-2\kappa_1^2}{(\nu^{n+1}-a)^2}\sum_{i\in L}\frac{(1-\theta)f_i}{(1+\theta)\kappa_1}(\kappa_i-\nu^{n+1})^2\frac{u_i^2}{u^2}\\
&= \frac{2\kappa_1}{\nu^{n+1}-a}\sum_{i\in L}f_i\frac{u_i^2}{u^2}(\kappa_i-\nu^{n+1})\\
&+\frac{4\theta}{1+\theta}\frac{\kappa_1}{(\nu^{n+1}-a)^2}\sum_{i\in L}(\kappa_i-\nu^{n+1})^2 f_i\frac{u_i^2}{u^2}\\
&-\frac{2\kappa_1}{(\nu^{n+1}-a)^2}\sum_{i\in L}f_i\frac{u_i^2}{u^2}(\kappa_i^2-(\nu^{n+1}+a)\kappa_i+a\nu^{n+1})\\
&\leq \frac{2\kappa_1}{\nu^{n+1}-a}\sum_{i\in L}f_i\frac{u_i^2}{u^2}(\kappa_i-\nu^{n+1})\\
&+\frac{4\theta}{1+\theta}\frac{\kappa_1}{(\nu^{n+1}-a)^2}\sum_{i\in L}(\kappa_i-\nu^{n+1})^2 f_i\frac{u_i^2}{u^2}+\frac{6\kappa_1}{a}F.
\end{aligned} \tag{7.23}$$

In deriving the last inequality in (7.23) we have used that $\kappa_i > 0$ for each i. Now fix θ so that $\frac{8\theta}{1+\theta} = a^2$, so we get the right hand side of (7.20) is strictly negative when provided $\kappa_1 > \frac{4}{a^2}$ which complete the proof. □

Let us assume that the flow exists in $[0, T)$ with $0 < T < \infty$ such that the norm of $u^2(t), \forall t \in [0, T)$ is uniformly bounded in $C^2(\Omega)$. Due to the concavity of F, we can apply the Evans-Krylov theorem [CC95] to get uniform $C^{2+\alpha}(\Omega)$ estimates which in turn will lead to $C^{2+\alpha,\frac{2+\alpha}{2}}(\Omega \times (0, T))$ estimates. And the long time existence follows by proving a priori estimates in any compact time interval for the corresponding norms.

In order to prove equation (1.16) in Theorem 1.2, according to Theorem 7.1 , we only need to find a uniform bound C which is independent of T for $u|D^2u|$ on the boundary $\partial\Omega \times [0, \infty)$.

Following Lemma 3.4 in [LX10], we obtain that, for any fixed $x \in \overline{\Omega}_\epsilon := \{x \in \overline{\Omega}, d(x, \partial\Omega) \leq \epsilon\}$,

$$u(x,t) - u(x,0) \leq \int_0^\infty uw(F-\sigma)dt = u(x,t^*)\int_0^\infty w(F-\sigma)dt \leq C\epsilon,$$

which implies that,

$$\int_0^\infty w(F-\sigma)dt \leq C \text{ in } \overline{\Omega}_\epsilon.$$

Therefore, by Lemma 4.1 and Corollary 5.4 we conclude that when $0 < \epsilon \leq \epsilon_0$, there exists a $\tilde{t}$ such that for any $t > \tilde{t}$, we have $0 \leq F - \sigma < \delta$ in $\overline{\Omega}_\epsilon$, where $\tilde{t}$ only depends on δ. Combining with Theorem 6.1 and Theorem 7.1 gives a uniform bound for $u|D^2u|$.

8. Convergence to a stationary solution

Let us go back to our original problem (1.13), which is a scalar parabolic differential equation defined on the cylinder $\Omega_T = \Omega \times [0,T)$ with initial value $u(0) = u_0 \in C^\infty(\Omega) \cap C^2(\overline{\Omega})$ and $u_0|_{\partial\Omega} = 0$. In view of the a priori estimates, which we have estimated in the preceding sections, we know that

$$u|D^2u| \leq C, \tag{8.1}$$

$$\sqrt{1+|Du|^2} \leq C, \tag{8.2}$$

and hence

$$F \text{ is uniformly elliptic in } u. \tag{8.3}$$

Moreover, since F is concave, we have uniform $C^{2+\alpha}(\Omega)$ estimates for $u^2(t),\ \forall t \geq 0$. Thus the flow exists for all $t \in [0,\infty)$.

By integrating equation (1.12) with respect to t, we get

$$u(x,t^*) - u(x,0) = \int_0^{t^*} (F-\sigma)uwdt. \tag{8.4}$$

In particular,

$$\int_0^\infty (F-\sigma)uwdt < \infty \ \ \forall x \in \Omega. \tag{8.5}$$

Hence for any $x \in \Omega$ there exists a sequence $t_k \to \infty$ such that $(F-\sigma)u(x,t_k) \to 0$.

On the other hand, $u(x,\cdot)$ is monotone increasing and bounded (see Lemma 3.3 of [LX10]). Therefore

$$\lim_{t\to\infty} u(x,t) = \tilde{u}(x) \tag{8.6}$$

exists, and is of class $C^\infty(\Omega) \cap C^1(\overline{\Omega})$. Moreover, $\tilde{u}(x)$ is a stationary solution of our problem, i.e., $F\left(\tilde{\Sigma}\right) = \sigma$, where $\tilde{\Sigma} = \{(x, \tilde{u}(x)) \mid x \in \Omega\}$.

Remark 8.1. Notice that without assumption (1.18) we may not have a unique stationary solution for equation (1.1), however due to our setting for the initial surface and the monotonicity of the height function $u(x,\cdot)$, we can see that the flow always converges uniformly to a unique stationary solution.

9. Uniqueness and foliation

Theorem 9.1. *Suppose f satisfies (1.3)-(1.9), in addition,*

$$\sum_i f_i > \sum_i \lambda_i^2 f_i \quad in\ K \cap \{0 < f < 1\}. \tag{9.1}$$

Let $\Sigma_i = \{(x, u_i(x) \mid x \in \Omega\},\ i = 1, 2,$ be two graphs such that

$$\sup_{x\in\Omega} f(\kappa[\Sigma_1]) < f(\kappa[\Sigma_2]), \tag{9.2}$$

where Σ_i $i = 1, 2$ are strictly locally convex graphs (oriented up) in $\mathbb{H}^{n+1}$ over $\Omega \subset \mathbb{R}^n$ with the same boundary Γ^ϵ in the horosphere $P_\epsilon = \{x_{n+1} = \epsilon\}$ or with the same asymptotic boundary $\Gamma = \partial\Omega$. Then there holds

$$u_1 > u_2, in\ \Omega. \tag{9.3}$$

Proof. We first observe that the weaker conclusion

$$u_1 \geq u_2 \tag{9.4}$$

is as good as the strict inequality (9.3), in view of the maximum principle.

Hence prove by contradiction, assume (9.4) is not valid, in another word,

$$E(u_2) = \{x \in \Omega : u_2(x) > u_1(x)\} \neq \emptyset. \tag{9.5}$$

Then there exists point $p_i \in \Sigma_i$ such that

$$0 < d_0 = d\,(\Sigma_1, \Sigma_2) = d(p_1, p_2) = \sup_{p\in\Sigma_1}\{\inf_{q\in\Sigma_2\cap I^+(\Sigma_1)} d(p, q) : (p, q) \in \Sigma_1 \times \Sigma_2\},$$

where d is the distance function in $\mathbb{R}^{n+1}$, and $I^+(\Sigma_1) = \{(x, x_{n+1}) : x^{n+1} \geq u_1(x)\}$.

Let χ be the maximal geodesic from Σ_1 to Σ_2 realizing this distance with end point p_1 and p_2, and parametrized by arc length. Denote by $\bar{d}$ the distance function to Σ_1,

$$\bar{d}(q) = \inf_{p\in\Sigma_1} d(p, q).$$

Since χ is maximal, $\Upsilon = \{\chi(t) : 0 \leq t < d_0\}$ contains no focal points of Σ_1, hence there exists an open neighborhood $\mathfrak{U} = \mathfrak{U}(\Upsilon)$ such that $\bar{d}$ is smooth in $\mathfrak{U}$, and $\mathfrak{U}$ is a tubular neighborhood of Σ_1, and hence covered by an associated normal Gaussian coordinates system (x^α) satisfying $x^{n+1} = \bar{d}$ in $\{x^{n+1} > 0\}$.

Now Σ_1 is the level set $\{\bar{d} = 0\}$, and the level set

$$\mathfrak{W}(s) = \{x \in \mathfrak{U} : \bar{d} = s\}$$

are smooth hypersurfaces. Since the principle curvatures of $\mathfrak{W}(t)$ at points along the normal geodesic emanating from any point of Σ_2 (say near p_2) are given by ode

$$\kappa_i'(s) = \kappa_i^2 - 1.$$

hence by (9.1) we have

$$\frac{d}{ds} f(\kappa)(s) = \sum \kappa_i^2 f_i - \sum f_i < 0 \ \text{ in } K \cap \{0 < f < 1\}. \tag{9.6}$$

Next, in the same way, we consider a tubular neighborhood $\mathfrak{N}$ of Σ_2 with corresponding normal Gaussian coordinates (x^α). The lever sets

$$\tilde{\mathfrak{W}}(r) = \{x^{n+1} = r\}, \ \ -\epsilon < r < 0,$$

lies below $\Sigma_2 = \tilde{\mathfrak{W}}(0)$ and are smooth for small ϵ.

Since the geodesic χ is perpendicular to Σ_2, it's also perpendicular to $\tilde{\mathfrak{W}}(r)$ and the length of the geodesic segment of χ is $-r$. Hence we deduce

$$d\left(\Sigma_1, \tilde{\mathfrak{W}}(r)\right) = d_0 + r.$$

Further more, for fixed r, the hypersurface $\tilde{\mathfrak{W}}(r)$ touches $\mathfrak{W}(d_0+r)$ at $p_r = \chi(d_0+r)$ from below. The maximum principle then implies

$$f|_{\tilde{\mathfrak{W}}(r)}(p_r) \leq f|_{\mathfrak{W}(d_0+r)}(p_r)$$

On the other hand, $\tilde{\mathfrak{W}}(r)$ converges to Σ_2. It follows from (9.6) that

$$f(\kappa[\Sigma_2])(\chi(d_0)) \leq f(\kappa[\Sigma_1])(\chi(0)),$$

which is a contradiction to (9.2). □

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References

[A94] B.Andrews, Contraction of convex hypersurfaces in Euclidean space, *Calc. Var.PDE* **2**, (1994), 151–171.

[CC95] L. Caffarelli, and X. Cabr, Fully nonlinear elliptic equations, *American Mathematical Society*,(1995).

[CNS84] L. Caffarelli, L. Nirenberg and J. Spruck, The Dirichlet problem for nonlinear second-order elliptic equations I, *Comm. Pure Applied Math.* **37** (1984), 369–402.

[CNS85] L. Caffarelli, L. Nirenberg and J. Sprick, The Dirichlet problem for nonlinear second-order elliptic equations III, *Acta Math.* **155** (1985), 261–301.

[CNS86] L. Caffarelli, L. Nirenberg and J. Sprick,Nonlinear second order elliptic equations IV, *Current Topics in P.D.E. Kinokunize Co.*, (1986),1–26.

[G96] C.Gerhardt, Closed Weigngarten hypersurfaces in Riemannian manifolds, *J.Differential Geom.***43**, (1996), 612–641.

[GS08] B. Guan, and J. Spruck, Hypersurfaces of constant curvature in Hyperbolic space II, preprint, arxiv.org/abs/0810.1781, *J. Eur. Math. Soc. to appear.*

[GSZ09] B. Guan, J. Spruck and M. Szapiel, Hypersurfaces of constant curvature in Hyperbolic space I, *J. Geom. Anal.* **19** (2009), no. 4, 772–795.

[GS10] B.Guan and J.Spruck, Convex hypersurfaces of Constant curvature in hyperbolic space, preprint.

[LX10] L. Lin and L. Xiao, Modified mean curvature flow of star-shaped hypersurfaces in hyperbolic space, preprint.